A NEW CONSTRUCTION OF THE SEMI-INFINITE BGG RESOLUTION

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1. Introduction

In their beautiful work [BGG] I. N. Bernshtein, I. M. Gelfand and S. I. Gelfand introduced so called Bernshtein-Gelfand-Gelfand resolutions of finite dimensional simple modules over semisimple Lie algebras consisting of Verma modules. Some years after that similar resolutions of integrable simple modules over general symmetrizable Kac-Moody algebras were constructed in [RW]. In both cases the resolutions, though obtained by purely algebraic methods, encoded geometric structure of the Flag manifolds of the corresponding Lie groups (the decomposition into the union of the Schubert cells).

In [FF] B. L. Feigin and E. Frenkel introduced some analogues of BGG resolutions of integrable simple modules over affine Kac-Moody algebras that consisted of Wakimoto modules. Mathematical folklore says that Wakimoto modules and Feigin-Frenkel resolutions represent some conjectural geometry of semi-infinite flag manifolds (see [FF]). This geometry should include in particular a semi-infinite analogue of the Schubert decomposition and a semi-infinite localization of a certain category of modules over the affine Lie algebra.

1.1. The present paper is devoted to an attempt of better understanding of the Feigin-Frenkel construction avoiding bosonization techniques and physical vocabulary used in [FF]. We construct a sequence of analogues of the affine BGG resolutions enumerated by elements of the affine Weyl group. Each of these complexes is quasiisomorphic to the integrable simple module over the affine Lie algebra. Moreover the complexes form an inductive system enumerated by the dominant chamber elements of the corresponding weight lattice and the limit of the system happens to be the semi-infinite BGG resolution of Feigin and Frenkel.

I hope that the techniques of the present paper will allow to construct semi-infinite BGG resolutions for general symmetrizable Kac-Moody algebras. It it possible also to perform an analogue of our construction for affine quantum groups for general q. This will be explained elsewhere.

1.2. Let us describe the structure of the paper. In the second section we develop the techniques of semiregular modules over Lie algebras. Namely let $\mathfrak{g}$ be a graded Lie algebra with a negatively graded finite dimensional subalgebra $\mathfrak{n}$. The $\mathfrak{g}$-module $S_{\mathfrak{n}} := \mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}} \mathrm{Coind}_{\mathbf{C}}^{\mathfrak{n}} \mathbf{C}$ is called the semiregular module over $\mathfrak{g}$. In the second and the third sections we present two different and independent proofs of the fact that the endomorphism algebra of $S_{\mathfrak{n}}$ contains the universal enveloping algebra of $\mathfrak{g}$. Thus $S_{\mathfrak{n}}$ becomes a $\mathfrak{g}$-bimodule. This allows us to construct a kernel functor $* \otimes_{U(\mathfrak{g})} S_{\mathfrak{n}} :\ \mathfrak{g}\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$.

In the fourth section we recall several facts from combinatorics of the affine root systems including definitions of the semi-infinite Bruhat order and the semi-infinite length function.

In the fifth section we apply the techniques from the second and the third sections in the case of affine Lie algebras. The twisting functors $* \otimes_{U(\mathfrak{g})} S_{\mathfrak{n}_w}$ for some finite dimensional nilpotent subalgebras $\mathfrak{n}_w$ enumerated by elements of the affine Weyl group are the affine analogues of the twisting functors defined in [FF] in the case of finite dimensional semisimple Lie algebras. We call the images of Verma modules under these functors the twisted Verma modules. Taking the images of the affine BGG resolutions under the twisting functors we obtain a family of complexes enumerated by elements of the affine Weyl group. We call these complexes the twisted BGG resolutions.

In the sixth section we show that twisted BGG resolutions form an inductive system and when the lengths of elements of the Weyl group tend to infinity, the corresponding twisted BGG resolutions "tend to the semi-infinite BGG resolution of Feigin and Frenkel".

2. Endomorphism algebra of the semiregular module

2.1. **Calculations in Clifford algebra.** Let $\mathfrak{n} = \bigoplus_{m \in Z_{<0}} \mathfrak{n}_m$ be a finite dimensional graded vector space, $\dim \mathfrak{n} = n$. Then there is a canonical bilinear form on the vector space $\mathfrak{n} \oplus \mathfrak{n}^*$:

$$\langle (n_1, f_1), (n_2, f_2) \rangle = f_2(n_1) + f_1(n_2),\ n_1, n_2 \in \mathfrak{n},\ f_1, f_2 \in \mathfrak{n}^*.$$

2.1.1. Recall that the *Clifford algebra* $C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)$ is a $\mathbf{C}$-algrbra with the space of generators equal to $\mathfrak{n} \oplus \mathfrak{n}^*$ and the following relations:

$$\omega_1 \omega_2 + \omega_2 \omega_1 = \langle \omega_1, \omega_2 \rangle,\ \omega_1, \omega_2 \in \mathfrak{n} \oplus \mathfrak{n}^*.$$

Remark: The subalgebra in $C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)$ generated by the space $\mathfrak{n}$ (resp. the space $\mathfrak{n}^*$) is isomorphic to the exterior algebra $\Lambda(\mathfrak{n})$ (resp. the exterior algebra $\Lambda(\mathfrak{n}^*)$).

2.1.2. **Lemma:** $C\ell(\mathfrak{n} \oplus \mathfrak{n}^*) \cong \Lambda(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*)$ as a vector space. □

In particular $\dim C\ell(\mathfrak{n} \oplus \mathfrak{n}^*) = 2^{2(\dim \mathfrak{n})}$. From now on the set $\{1, \ldots, n\}$ is denoted by $\underline{n}$. Choose a homogeneous base $\{e_i | i \in \underline{n}\}$ of the vector space $\mathfrak{n}$ and its dual base $\{e_i^* | i \in \underline{n}\} \subset \mathfrak{n}^*$. Fix a subset $I \subset \{1, \ldots, \dim \mathfrak{n}\}$ and denote its complementary set by $\overline{I}$. Denote the element of the Clifford algebra $\prod_{i \in I} e_i$ (resp. the element $\prod_{i \in I} e_i^*$) by e_I (resp. by e_I^*). The order of factors in the monomials is the one in the set $\underline{n}$.

2.1.3. Consider the left $C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)$-modules

$$\underline{\Lambda(\mathfrak{n})} := \mathrm{Ind}_{\Lambda(\mathfrak{n}^*)}^{C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)} \underline{\mathbf{C}} \text{ and } \underline{\Lambda(\mathfrak{n}^*)} := \mathrm{Ind}_{\Lambda(\mathfrak{n})}^{C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)} \underline{\mathbf{C}}.$$

Here $\underline{\mathbf{C}}$ denotes the trivial module. Then the images of the elements $\{e_I | I \subset \underline{n}\}$ (resp. of the elements $\{e_I^* | I \subset \underline{n}\}$) form a base in the space $\underline{\Lambda(\mathfrak{n})}$ (resp. $\underline{\Lambda(\mathfrak{n}^*)}$). We are going to calculate the action of the Clifford algebra in these bases explicitly. Denote the matrix element $e_I \mapsto e_J$ (resp. $e_I^* \mapsto e_J^*$) by a_I^J (resp. by b_I^J).

Consider the elements in the Clifford algebra of the form $e_J e_{\underline{n}}^* e_I,\ I, J \subset \underline{n}$.

2.1.4. **Lemma:**

(i) The element $e_J e_{\underline{n}}^* e_I$ acts as $a_{\overline{I}}^J :\ \underline{\Lambda(\mathfrak{n})} \longrightarrow \underline{\Lambda(\mathfrak{n})}$;

(ii) The element $e_J e_{\underline{n}}^* e_I$ acts as $b_I^{\overline{J}} :\ \underline{\Lambda(\mathfrak{n}^*)} \longrightarrow \underline{\Lambda(\mathfrak{n}^*)}$. □

2.1.5. **Corollary:** In particular $C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)$ is isomorphic to the $(2^n \times 2^n)$ matrix algebra. □

2.2. The antiautomorphism of the tensor algebra $T(\mathfrak{n} \oplus \mathfrak{n}^*)$

$$\sigma :\ a_1 \otimes \ldots \otimes a_k \mapsto \mathfrak{a}_k \otimes \ldots \otimes a_1,\ a_1, \ldots, a_k \in \mathfrak{n} \oplus \mathfrak{n}^*.$$

preserves the relation defining the Clifford algebra, hence the restriction of σ,

$$\sigma :\ C\ell(\mathfrak{n} \oplus \mathfrak{n}^*) \longrightarrow C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)^{\mathrm{opp}},$$

is correctly defined. The antiautomorphism σ preserves the subalgebras $\Lambda(\mathfrak{n})$ and $\Lambda(\mathfrak{n}^*)$. For any $\omega \in \Lambda^k(\mathfrak{n})$ or $\omega \in \Lambda^k(\mathfrak{n}^*)$ we have $\sigma(\omega) = (-1)^{\left[\frac{k}{2}\right]} \omega$, where $[\cdot]$ denotes the integral part of the number.

2.2.1. **Remark:** There is a natural isomorphism of algebras

$$\beta :\ \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n})}) \widetilde{\longrightarrow} \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n}^*)})^{\mathrm{opp}},\ a_I^J \mapsto b_J^I,\ I, J \subset \underline{n}.$$

Denote the isomorphism $C\ell(\mathfrak{n} \oplus \mathfrak{n}^*) \widetilde{\longrightarrow} \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n})})$ by α, the isomorphism $\mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n}^*)}) \widetilde{\longrightarrow} C\ell(\mathfrak{n} \oplus \mathfrak{n}^*)$ is denoted by γ.

2.2.2. **Lemma:** $\gamma \circ \beta \circ \alpha = \sigma$.

Proof. Follows immediately from 2.1.4. □

2.3. **Endomorphism algebra of the semiregular module.** Suppose we have a graded Lie algebra $\mathfrak{g} = \bigoplus_{m \in \mathbf{Z}} \mathfrak{g}_m$ and a negatively graded finite dimensional Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$, $\dim \mathfrak{n} = n$. Note that $\mathfrak{n}$ is automatically nilpotent. Suppose also that $\mathfrak{n}$ acts locally ad-nilpotently on $\mathfrak{g}$ (and thus on $U(\mathfrak{g})$ as well). Fix a graded vector subspace $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$ as a vector space. Denote the $\mathfrak{g}$-module $\mathrm{Ind}_{U(\mathfrak{n})}^{U(\mathfrak{g})} \underline{\mathbf{C}}$ by $\underline{U(\mathfrak{b})}$. Choose a homogeneous base $\{e_i\}_{i \in \underline{n}}$ of $\mathfrak{n}$. Thus the bracket in $\mathfrak{n}$ is defined by the structure constants: $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$.

2.3.1. The category of graded left $\mathfrak{g}$-modules $M = \bigoplus_{m \in \mathbf{Z}} M_m$ with morphisms being $\mathfrak{g}$-module homomorphisms that preserve gradings is denoted by $\mathfrak{g}$-mod. The corresponding category of $\mathfrak{n}$-modules is denoted by $\mathfrak{n}$-mod. Let $\langle \cdot \rangle$ denote the grading shift functor:

$$\langle \cdot \rangle : \ \mathfrak{g}\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}, \ M \in \mathfrak{g}\text{-mod}, \text{ then } M\langle k \rangle_m := M_{k+m}.$$

Denote the space $\bigoplus_{k \in \mathbf{Z}} \mathrm{Hom}_{\mathfrak{g}\text{-mod}}(M_1, M_2\langle k \rangle)$ by $\mathrm{Hom}_{U(\mathfrak{g})}(M_1, M_2)$. A similar notation is used in the category of $\mathfrak{n}$-modules.

2.3.2. The grading on $\mathfrak{n}$ induces a natural grading on its universal enveloping algebra $U(\mathfrak{n})$. Consider a *left* coregular $\mathfrak{n}$-module $U(\mathfrak{n})^* \in \mathfrak{n}$-mod:

$$U(\mathfrak{n})^* := \bigoplus_{m \in \mathbf{N}} \mathrm{Hom}_{\mathbf{C}}(U(\mathfrak{n})_{-m}, \mathbf{C}).$$

The left action of $\mathfrak{n}$ is defined as follows:

$$f : \ U(\mathfrak{n}) \longrightarrow \mathbf{C}, \ g \in \mathfrak{n}, \text{ then } (n \cdot f)(u) := f(un), \ u \in U(\mathfrak{n}).$$

Definition: The left $\mathfrak{g}$-module $S_{\mathfrak{n}} := \mathrm{Ind}_{U(\mathfrak{n})}^{U(\mathfrak{g})} U(\mathfrak{n})^*$ is called the left semiregular representation of $\mathfrak{g}$ with respect to the subalgebra $\mathfrak{n}$.

Our main task in this section is to find the endomorphism algebra $\mathrm{End}_{U(\mathfrak{g})}(S_{\mathfrak{n}})$. Let us calculate it first as a vector space.

2.3.3. **Lemma:** $\mathrm{Ext}^{>0}_{\mathfrak{g}\text{-mod}}(S_{\mathfrak{n}}, S_{\mathfrak{n}}) = 0, \ \mathrm{End}_{U(\mathfrak{g})}(S_{\mathfrak{n}}) \cong \mathrm{Hom}_{\mathbf{C}}(U(\mathfrak{n})^*, \underline{U(\mathfrak{b})})$.

Proof. By Schapiro lemma $\mathrm{Ext}^{\bullet}_{U(\mathfrak{g})}(S_{\mathfrak{n}}, S_{\mathfrak{n}}) = \mathrm{Ext}^{\bullet}_{U(\mathfrak{n})}(U(\mathfrak{n})^*, S_{\mathfrak{n}})$. But the $U(\mathfrak{n})$-module $S_{\mathfrak{n}}$ is cofree with the space of cogenerators equal to $\underline{U(\mathfrak{b})}$. Again use Schapiro lemma. □

2.3.4. **Remark:** Thus we see that up to a completion the space $\mathrm{End}_{U(\mathfrak{g})}(S_{\mathfrak{n}})$ is equal to $U(\mathfrak{n}) \otimes \underline{U(\mathfrak{b})} \cong U(\mathfrak{g})$.

The considerations below remain true also for subalgebras $\mathfrak{n} \subset \mathfrak{g}$ such that the adjoint action of $\mathfrak{n}$ on $\mathfrak{g}$ is not locally nilpotent. In this case the algebra that is being calculated equals to the endomorphism algebra of some completion of $S_{\mathfrak{n}}$.

2.4. Consider the $U(\mathfrak{g})$-free resolution $P^{\bullet}(S_{\mathfrak{n}})$ of $S_{\mathfrak{n}}$ as follows:

$$P^{-k}(S_{\mathfrak{n}}) := U(\mathfrak{g}) \otimes \Lambda^k(\mathfrak{n}) \otimes U(\mathfrak{n})^*, \ d : \ P^{-k}(S_{\mathfrak{n}}) \longrightarrow P^{-k+1}(S_{\mathfrak{n}}),$$
$$u \in U(\mathfrak{g}), f \in U(\mathfrak{n})^*, \ n_1, \dots, n_k \in \mathfrak{n}, \ d(u \otimes n_1 \wedge \dots \wedge n_k \otimes f) :=$$
$$\sum_{j=1}^k (-1)^j (un_j \otimes n_1 \wedge \dots \wedge n_{j-1} \wedge n_{j+1} \wedge \dots \wedge n_k \otimes f + u \otimes n_1 \wedge \dots \wedge n_{j-1} \wedge n_{j+1} \wedge \dots \wedge n_k \otimes (n_j \cdot f))$$
$$+ \sum_{i<j} (-1)^{i+j} u \otimes [n_i, n_j] \wedge n_1 \wedge \dots \wedge n_{i-1} \wedge n_{i+1} \wedge \dots \wedge n_{j-1} \wedge n_{j+1} \wedge \dots \wedge n_k \otimes f.$$

2.4.1. Consider a graded subalgebra $A^{\bullet} \subset \mathrm{End}_{U(\mathfrak{g})}(P^{\bullet}(S_{\mathfrak{n}}))$:

$$A^{\bullet} := U(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{C}}(\Lambda(\mathfrak{n}) \otimes U(\mathfrak{n})^*) \widetilde{\longrightarrow} U(\mathfrak{g}) \otimes \mathcal{C}\ell(\mathfrak{n} \oplus \mathfrak{n}^*) \otimes \mathrm{End}_{\mathbf{C}}(U(\mathfrak{n})^*).$$

2.4.2. **Lemma:** $A^\bullet$ is a DG-subalgebra in $\mathrm{End}_{U(\mathfrak{g})}(P^\bullet(S_\mathfrak{n}))$. $\square$

Denote the elements of the chosen base of $\mathfrak{n}$ (resp. of $\mathfrak{n}^*$) considered as elements of $\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)$ by $\{\overline{e}_i\}_{i\in\underline{n}}$, (resp. by $\{\overline{e}_i^*\}_{i\in\underline{n}}$). Consider the element in $A^\bullet$ as follows:

$$D_{A^\bullet} := \sum_{j=1}^{k}(e_i\otimes\overline{e}_i^*\otimes 1 + 1\otimes\overline{e}_i^*\otimes l_{e_i}) + \sum_{i<j,k} c_{i,j}^k\overline{e}_i^*\overline{e}_j^*\overline{e}_k,$$

where $l_{e_i}(f)$ denotes the element $(e_i\cdot f)\in U(\mathfrak{n})^*$.

2.4.3. **Lemma:** The differential in the DG-algebra $A^\bullet$ is given by the supercommutator with the element $D_{A^\bullet}$: for every $a\in A^\bullet$ we have $d(a)=\{D_{A^\bullet},a\}$. $\square$

2.4.4. **Lemma:** $H^{\neq 0}(A^\bullet)=0$. $\square$

2.5. We construct another DG-algebra of the same size. Consider the left regular $U(\mathfrak{g})$-module $U(\mathfrak{g})$ and its $U(\mathfrak{g})$-free resolution $Q^\bullet(U(\mathfrak{g}))$:

$$Q^{-k}(U(\mathfrak{g})) := U(\mathfrak{g})\otimes U(\mathfrak{n})\otimes\Lambda^k(\mathfrak{n}),\ d:\ Q^{-k}(U(\mathfrak{g}))\longrightarrow Q^{-k+1}(U(\mathfrak{g})),$$
$$u\in U(\mathfrak{g}), m\in U(\mathfrak{n}),\ n_1,\dots,n_k\in\mathfrak{n},\ d(u\otimes m\otimes n_1\wedge\dots\wedge n_k) :=$$
$$\sum_{j=1}^{k}(-1)^j(un_j\otimes m\otimes n_1\wedge\dots\wedge n_{j-1}\wedge n_{j+1}\wedge\dots\wedge n_k + u\otimes mn_j\otimes n_1\wedge\dots\wedge n_{j-1}\wedge n_{j+1}\wedge\dots\wedge n_k)$$
$$+\sum_{i<j}(-1)^{i+j}u\otimes m\otimes[n_i,n_j]\wedge n_1\wedge\dots\wedge n_{i-1}\wedge n_{i+1}\wedge\dots\wedge n_{j-1}\wedge n_{j+1}\wedge\dots\wedge n_k$$

(this seems to be unnessesary since $U(\mathfrak{g})$ is free over itself).

2.5.1. Consider a graded subalgebra $B^\bullet\subset\mathrm{End}_{U(\mathfrak{g})}(Q^\bullet(U(\mathfrak{g})))$:

$$B^\bullet := U(\mathfrak{g})\otimes\mathrm{End}_{\mathbf{C}}(\Lambda(\mathfrak{n})\otimes U(\mathfrak{n}))\widetilde{\longrightarrow}U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)\otimes\mathrm{End}_{\mathbf{C}}(U(\mathfrak{n})).$$

and an element $D_{B^\bullet}$ in $B^\bullet$ as follows:

$$D_{B^\bullet} := \sum_{j=1}^{k}(e_i\otimes\overline{e}_i^*\otimes 1 + 1\otimes\overline{e}_i^*\otimes r_{e_i}) + \sum_{i<j,k} c_{i,j}^k\overline{e}_i^*\overline{e}_j^*\overline{e}_k,$$

where $r_{e_i}(m)$ denotes the element $(m\cdot e_i)\in U(\mathfrak{n})$.

2.5.2. **Lemma:** The differential in the DG-algebra $B^\bullet$ is given by the supercommutator with the element $D_{B^\bullet}$: for every $b\in B^\bullet$ we have $d(b)=\{D_{B^\bullet},b\}$. $\square$

2.5.3. **Lemma:** $H^{\neq 0}(B^\bullet)=0$. $\square$

We construct an inclusion ι of the algebra $U(\mathfrak{g})$ into B^0 such that $\iota:\ U(\mathfrak{g})\hookrightarrow B^\bullet$ is a quasiisomorphism of DG-algebras.

2.5.4. **Lemma:** For any right $\mathfrak{n}$-module X consider the right $\mathfrak{n}$-module $X_1 := X\otimes U(\mathfrak{n})$ with $\mathfrak{n}$-action provided by the Hopf algebra structure on $U(\mathfrak{n})$ and the right $\mathfrak{n}$-module $X_2 := X\otimes U(\mathfrak{n})$ with $\mathfrak{n}$-action trivial on the first factor. Then $X_1\widetilde{\longrightarrow}X_2$.

Proof. Consider the comultiplication map in the algebra $U(\mathfrak{n})$:

$$\Delta:\ U(\mathfrak{n})\longrightarrow U(\mathfrak{n})\otimes U(\mathfrak{n}),\ \Delta(u)=\sum_i\Delta_i^{(1)}(u)\otimes\Delta_i^{(2)}(u)+1\otimes u,\ u\in U(\mathfrak{n}).$$

We introduce a map $\phi:\ X_2\longrightarrow X_1$ as follows:

$$\phi:\ x\otimes u\mapsto x\otimes u+\sum_i(x\cdot\Delta_i^{(1)}(u))\otimes\Delta_i^{(2)}(u),\ x\in X,\ u\in U(\mathfrak{n}).$$

Here $\cdot$ denotes the right $\mathfrak{n}$-action on X. Evidently ϕ is an isomorphism of vector spaces. One checks directly that ϕ commutes with the defined $\mathfrak{n}$-actions on X_1 and X_2. $\square$

Let $\widetilde{U(\mathfrak{g})\otimes U}(\mathfrak{n})$ be the $U(\mathfrak{g})$-$U(\mathfrak{n})$ bimodule with left $U(\mathfrak{g})$-action via the first factor and right $U(\mathfrak{n})$-action via the second one. Consider also the $U(\mathfrak{g})$-$U(\mathfrak{n})$ bimodule $\overline{U(\mathfrak{g})\otimes U(\mathfrak{n})}$ with left $U(\mathfrak{g})$-action via the first factor and right $U(\mathfrak{n})$-action using the comultiplication on $U(\mathfrak{n})$.

2.5.5. **Corollary:** $\phi:\ \widetilde{U(\mathfrak{g})\otimes U}(\mathfrak{n})\widetilde{\longrightarrow}\overline{U(\mathfrak{g})\otimes U(\mathfrak{n})}$. □

Remark: Note that $Q^\bullet(U(\mathfrak{g}))$ is the standard complex for the computation of the Lie algebra homology of $\mathfrak{n}$ with coefficients in the right module $\overline{U(\mathfrak{g})\otimes U(\mathfrak{n})}$.

Note also that the right $U(\mathfrak{g})$-action on $\widetilde{U(\mathfrak{g})\otimes U}(\mathfrak{n})$ commutes with both the left $U(\mathfrak{g})$-module and the right $U(\mathfrak{n})$-module structures. Thus we obtain the following statement.

2.5.6. **Lemma:** The map $g\mapsto \phi\circ(r_g\otimes 1\otimes 1)\circ\phi^{-1}$, where r_g denotes the right multiplication by g, defines a DG-algebra homomorphism $U(\mathfrak{g})\longrightarrow \mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g})\otimes U(\mathfrak{n})\otimes\Lambda^\bullet(\mathfrak{n}))=B^\bullet$ that becomes an isomorphism on cohomologies.

Proof. Follows immediately from the previous Remark. □

2.5.7. **Remark:** Clearly the map from the previous Lemma is a composition

$$U(\mathfrak{g})\xrightarrow{\iota} B^\bullet\longrightarrow \mathrm{End}_{U(\mathfrak{g})}(Q^\bullet(U(\mathfrak{g}))).$$

Our main result in this section is the following statement. Consider the following isomorphisms of graded algebras (not nessesarily preserving differentials):

$$\theta:\ B^{\bullet\mathrm{opp}}=(U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)\otimes\mathrm{End}_{\mathbf{C}}(U(\mathfrak{n})))^{\mathrm{opp}}\widetilde{\longrightarrow}U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)\otimes(\mathrm{End}_{\mathbf{C}}(U(\mathfrak{n})))^{\mathrm{opp}}$$
$$\theta=\sigma_{\mathfrak{g}}\otimes\sigma_{\mathfrak{n}},\ \sigma_{\mathfrak{g}}(g)=-g,\ g\in\mathfrak{g},\ \sigma(\overline{e}_i)=\overline{e}_i,\ \sigma(\overline{e}_i^*)=\overline{e}_i^*\ \ i\in\underline{n}$$
$$\eta:\ U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)\otimes(\mathrm{End}_{\mathbf{C}}(U(\mathfrak{n})))^{\mathrm{opp}}\widetilde{\longrightarrow}U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)\otimes\mathrm{End}_{\mathbf{C}}(U(\mathfrak{n})^*)=A^\bullet.$$

2.5.8. **Theorem:** The map $\eta\circ\theta:\ B^{\bullet\mathrm{opp}}\longrightarrow A^\bullet$ is homomorphism of DG-algebras.

Remark: In particular there is a canonical inclusion of algebras $U(\mathfrak{g})\longrightarrow\mathrm{End}_{U(\mathfrak{g})}(S_{\mathfrak{n}})$ that is an isomorphism up to a certain completion of $\mathrm{Im}\,U(\mathfrak{g})$.

2.6. **Proof of the Theorem 2.5.8.** We begin the proof with presenting several Lemmas.

2.6.1. **Remark:** Let $\mathfrak{n}$ be an arbitrary Lie algebra with a base $\{e_i\}_{i\in\overline{n}}$ and structure constants $c_{i,j}^k,\ i,j,k\in\underline{n}$. Then for a fixed $k\in\underline{n}$ we have $\sum_{i,j\in\overline{n}}c_{i,j}^k=0$.

Let $D_{A^\bullet}^{(1)}$ and $D_{B^\bullet}^{(1)}$ be the summands of $D_{A^\bullet}$ and $D_{B^\bullet}$ respectively that belong to $U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)$ as follows:

$$D_{A^\bullet}^{(1)}:=\sum_{j=1}^{k}e_i\otimes\overline{e}_i^*\otimes 1+\sum_{i<j,k}c_{i,j}^k\overline{e}_i^*\overline{e}_j^*\overline{e}_k,$$

and $D_{B^\bullet}^{(1)}\in B^\bullet$ given by the same formula.

2.6.2. **Lemma:**

$$\sigma(D_{B^\bullet}^{(1)})=-\sum_{j=1}^{k}e_i\otimes\overline{e}_i^*\otimes 1-\sum_{i<j,k}c_{i,j}^k\overline{e}_i^*\overline{e}_j^*\overline{e}_k,$$

in particular $d^{(1)}:\ U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)\longrightarrow U(\mathfrak{g})\otimes\mathcal{C}\ell(\mathfrak{n}\oplus\mathfrak{n}^*)$, $d^{(1)}(\cdot):=\{D_{B^\bullet}^{(1)},\cdot\}$, commutes with σ.

Proof. The only thing to be checked is that

$$\sigma_{\mathfrak{n}}\left(\sum_{i<j,k}c_{i,j}^k\overline{e}_i^*\overline{e}_j^*\overline{e}_k\right)=-\sum_{i<j,k}c_{i,j}^k\overline{e}_i^*\overline{e}_j^*\overline{e}_k.$$

The calculation looks as follows:

$$\sigma_{\mathfrak{n}}\left(\sum_{i<j,k} c_{i,j}^k \overline{e}_i^* \overline{e}_j^* \overline{e}_k\right) = \sum_{i<j,k} c_{i,j}^k \overline{e}_k \overline{e}_j^* \overline{e}_i^* = -\sum_{i<j,k} c_{i,j}^k \overline{e}_j^* \overline{e}_k \overline{e}_i^* + \sum_{i<j} c_{i,j}^j \overline{e}_i^* =$$
$$-\sum_{i<j,k} c_{i,j}^k \overline{e}_i^* \overline{e}_j^* \overline{e}_k + \sum_{i<j} c_{i,j}^j \overline{e}_i^* - \sum_{i<j} c_{i,j}^i \overline{e}_j^* =$$
$$-\sum_{i<j,k} c_{i,j}^k \overline{e}_i^* \overline{e}_j^* \overline{e}_k + \sum_{i<j} c_{i,j}^j \overline{e}_i^* + \sum_{i>j} c_{i,j}^j \overline{e}_i^*.$$

The statement follows from Remark 2.6.1. □

Let $D_{A^\bullet}^{(2)} := D_{A^\bullet} - D_{A^\bullet}^{(1)}$ and $D_{B^\bullet}^{(2)} := D_{B^\bullet} - D_{B^\bullet}^{(1)}$. It remains to prove that $\eta \circ \theta(D_{B^\bullet}^{(2)}) = -D_{A^\bullet}^{(2)}$.

For a left $\mathfrak{n}$-module X consider the $\mathcal{C}\ell(\mathfrak{n} \oplus \mathfrak{n}^*) \otimes \mathrm{End}_{\mathbf{C}}(X)$-modules $\underline{\Lambda(\mathfrak{n})} \otimes X$ and $\underline{\Lambda(\mathfrak{n}^*)} \otimes X$. The corresponding identifications

$$\mathcal{C}\ell(\mathfrak{n} \oplus \mathfrak{n}^*) \otimes \mathrm{End}_{\mathbf{C}}(X) \cong \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n})} \otimes X) \text{ and } \mathcal{C}\ell(\mathfrak{n} \oplus \mathfrak{n}^*) \otimes \mathrm{End}_{\mathbf{C}}(X) \cong \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n}^*)} \otimes X)$$

are denoted by φ and ψ. Consider the map $d : \underline{\Lambda(\mathfrak{n})} \otimes X \longrightarrow \underline{\Lambda(\mathfrak{n})} \otimes X$,

$$a_1 \wedge \ldots \wedge a_k \otimes x \mapsto \sum_{j=1}^{k} (-1)^j a_1 \wedge \ldots \wedge a_{j-1} \wedge a_{j+1} \wedge \ldots \wedge a_k \otimes a_j \cdot x,$$

(resp. the map $\overline{d}^* : \underline{\Lambda(\mathfrak{n}^*)} \otimes X \longrightarrow \underline{\Lambda(\mathfrak{n}^*)} \otimes X$, $a_1^* \wedge \ldots \wedge a_k^* \otimes x \mapsto \sum_{s \in \underline{n}} \overline{e}_s^* \wedge a_1^* \wedge \ldots \wedge a_k^* \otimes e_s \cdot x$).

2.6.3. **Lemma:** $\varphi^{-1}(d) = \psi^{-1}(\overline{d}^*)$.

Proof. Follows from 2.1.4 □

Now by 2.2.2 the map $\eta \circ \theta$ can be viewed as follows:

$$B^{\bullet\mathrm{opp}} = \Big(U(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n})} \otimes U(\mathfrak{n}))\Big)^{\mathrm{opp}} \widetilde{\longrightarrow} U(\mathfrak{g}) \otimes \Big(\mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n})} \otimes U(\mathfrak{n}))\Big)^{\mathrm{opp}}$$
$$\widetilde{\longrightarrow} U(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n}^*)} \otimes U(\mathfrak{n})^*) \widetilde{\longrightarrow} U(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n})} \otimes U(\mathfrak{n})^*) = A^\bullet.$$

Here the map $U(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n}^*)} \otimes U(\mathfrak{n})^*) \widetilde{\longrightarrow} U(\mathfrak{g}) \otimes \mathrm{End}_{\mathbf{C}}(\underline{\Lambda(\mathfrak{n})} \otimes U(\mathfrak{n})^*)$ is given by $\varphi \circ \psi^{-1}$ for $X = U(\mathfrak{n})^*$. Note that the map d in the previous Lemma is given by the supercommutator with $D_{A^\bullet}^{(2)}$, and the map $\overline{d}^*$ is given by the supercommutator with the image of $D_{B^\bullet}^{(2)}$. So the theorem is proved. □

Remark: In fact we did not use the Lie algebra structure on $\mathfrak{g}$ in the proof of Theorem 2.5.8. So the statement of the Theorem holds in the following situation. Let C be a graded associative algebra containing a finite dimensional negatively graded nilpotent Lie subalgebra $\mathfrak{n}$ that acts locally ad-nilpotently on C. Suppose that C contains $U(\mathfrak{n})$ as a graded subalgebra and is free both as a left and a right $U(\mathfrak{n})$-module. Suppose also there exists an antipode map $\alpha : C \longrightarrow C^{\mathrm{opp}}$ that preserves $\mathfrak{n} \subset C$. Consider the left semiregular C-module $S_{\mathfrak{n}} := C \otimes_{U(\mathfrak{n})} U(\mathfrak{n})^*$. Then there exists an inclusion of associative algebras

$$C \longrightarrow \mathrm{End}_C(S_{\mathfrak{n}}).$$

3. Endomorphisms of the semiregular module (continued)

In this section we present another proof of Theorem 2.5.8. Like in the previous section let $\mathfrak{g} = \bigoplus_{m \in \mathbf{Z}} \mathfrak{g}_m$ be a graded Lie algebra with a finite dimensional negatively graded Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$ that acts locally ad-nilpotently both on $\mathfrak{g}$ and on $U(\mathfrak{g})$.

3.1. Suppose $\mathfrak{n}$ is abelian, and $\dim \mathfrak{n} = n$. Choose a homogenious base $\{e_i | i \in \underline{n}\}$ in $\mathfrak{n}$. Let $\{x_i | i \in \underline{n}\}$ be the dual base in $\mathfrak{n}^*$. Then $U(\mathfrak{n}) = \mathbf{C}[e_i | i \in \underline{n}]$ and $U(\mathfrak{n})^* = \mathbf{C}[x_i, i \in \underline{n}]$. The Lie algebra $\mathfrak{n}$ acts on $U(\mathfrak{n})^*$ by derivations: e_i acts as $\partial/\partial x_i$.

3.1.1. Consider the left $\mathfrak{g} \oplus \mathfrak{n}$-module $U(\mathfrak{g}) \otimes U(\mathfrak{n})^* \cong U(\mathfrak{g}) \otimes \mathbf{C}[x_i | i \in \underline{n}]$. For every $u \in U(\mathfrak{g})$ consider an endomorphism $\sigma(u)$ of $U(\mathfrak{g}) \otimes U(\mathfrak{n})^*$ as follows:

$$\sigma(u)(v \otimes p) := v \exp\left(\sum_{i=1}^{n} \mathrm{ad}_{e_i} \otimes x_i\right)(u)p, \ v \in U(\mathfrak{g}), \ p \in \mathbf{C}[x_i | i \in \underline{n}].$$

3.1.2. **Lemma:** For every $u \in U(\mathfrak{g})$ the endomorphism $\sigma(u)$ commutes both with the diagonal action of $U(\mathfrak{n})$ and with the left regular action of $U(\mathfrak{g})$ on $U(\mathfrak{g}) \otimes U(\mathfrak{n})^*$.

Proof. A direct calculation. $\square$

Corollary: σ defines an inclusion of algebras $U(\mathfrak{g}) \hookrightarrow \mathrm{End}_{\mathfrak{g}}(S_{\mathfrak{n}})$. $\square$

3.2. Let $\mathfrak{g}$ and $\mathfrak{n}$ be like in the beginning of this section. Then there exists a filtration F on $\mathfrak{n}$:

$$\mathfrak{n} = F^0\mathfrak{n} \supset F^1\mathfrak{n} \supset \ldots \supset F^{\mathrm{top}}\mathfrak{n} = 0$$

such that each $F^i\mathfrak{n}$ is an ideal in $F^{i-1}\mathfrak{n}$ and there exist abelian subalgebras $\mathfrak{n}^i \subset F^i\mathfrak{n}$ such that each $F^i\mathfrak{n} = \mathfrak{n}^i \oplus F^{i+1}\mathfrak{n}$ as a vector space.

3.2.1. **Theorem:** There exists an inclusion of algebras $U(\mathfrak{g}) \hookrightarrow \mathrm{End}_{\mathfrak{g}}(S_{\mathfrak{n}})$ such that the image of $U(\mathfrak{g})$ is a dense subalgebra in $\mathrm{End}_{\mathfrak{g}}(S_{\mathfrak{n}})$.

3.2.2. **Remark:** The rest of this subsection is devoted to the proof of Theorem 3.2.1.

Suppose $\mathfrak{g} \supset \mathfrak{n} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$ as a graded vector space, where $\mathfrak{n}^-$ is an ideal in $\mathfrak{n}$ and $\mathfrak{n}^+$ is a subalgebra in $\mathfrak{n}$. Consider the left $\mathfrak{n}$-modules

$$S_{\mathfrak{n}^-}^{\mathfrak{n}} := U(\mathfrak{n}) \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{n}^-)^* \text{ and } S_{\mathfrak{n}^+}^{\mathfrak{n}} := U(\mathfrak{n}) \otimes_{U(\mathfrak{n}^+)} U(\mathfrak{n}^+)^*.$$

Then the $\mathfrak{g}$-modules $S_{\mathfrak{n}^-}$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} S_{\mathfrak{n}^-}^{\mathfrak{n}}$ (resp. the $\mathfrak{g}$-modules $S_{\mathfrak{n}^+}$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} S_{\mathfrak{n}^+}^{\mathfrak{n}}$) are naturally isomorphic.

The induction functor $\mathrm{Ind}_{\mathfrak{n}^-}^{\mathfrak{n}} : \mathfrak{n}^-\text{-mod} \longrightarrow \mathfrak{n}\text{-mod}$ takes $U(\mathfrak{n}^-)^*$ to $S_{\mathfrak{n}^-}^{\mathfrak{n}}$ and provides an inclusion of algebras $U(\mathfrak{n}) \subset \mathrm{End}_{\mathfrak{n}}(S_{\mathfrak{n}^-}^{\mathfrak{n}})$:

$$n \cdot a \otimes f := a \otimes n \cdot f, \text{ where } n \in \mathfrak{n}^-, f \in U(\mathfrak{n}^-)^*, a \in U(\mathfrak{n}), \ n \cdot f(n') := f(n'n), \mathfrak{n}' \in \mathfrak{n}^-.$$

3.2.3. Consider the following action of $\mathfrak{n}^+$ on $U(\mathfrak{n}) \otimes U(\mathfrak{n}^-)^*$:

$$n^+ \cdot a \otimes f := an^+ \otimes f + a \otimes [n^+, f], \text{ where } n^+ \in \mathfrak{n}^+, f \in U(\mathfrak{n}^-)^*, a \in U(\mathfrak{n}), \ [n^+, f](u) := f([u, n^+]), \ u \in U(\mathfrak{n}^-).$$

3.2.4. **Lemma:** The defined action of $\mathfrak{n}^+$ on $U(\mathfrak{n}) \otimes U(\mathfrak{n}^-)^*$ commutes with the left regular action of $U(\mathfrak{n})$. Moreover it is well defined on $S_{\mathfrak{n}^-}^{\mathfrak{n}}$. Along with the action of $\mathfrak{n}^-$ on $S_{\mathfrak{n}^-}^{\mathfrak{n}}$ it defines the inclusion of algebras $U(\mathfrak{n}) \subset \mathrm{End}_{\mathfrak{n}}(S_{\mathfrak{n}^-}^{\mathfrak{n}})$.

Proof. The first statement is obvious. Let $a \in U(\mathfrak{n})$, $f \in U(\mathfrak{n}^-)^*$, $n^+ \in \mathfrak{n}^+$, $n^- \in \mathfrak{n}^-$. Note that $[n^-, n^+] \in \mathfrak{n}^-$. Thus we have

$$n^+ \cdot an^- \otimes f = an^-n^+ \otimes f + a^n - \otimes [n^+, f] = an^-n^+ \otimes f + a \otimes n^-[n^+, f]$$
$$= an^+n^- \otimes f + a \otimes [n^-, n^+]f - a \otimes [n^-, n^+]f + a \otimes [n^+, n^-f] = n^+ \cdot a \otimes n^-f.$$

To prove the third statement note that

$$n^+ \cdot n^- \cdot a \otimes f = an^+ \otimes n^-f + a \otimes [n^+, n^-f]$$
$$= an^+ \otimes n^-f + a \otimes n^-[n^+, f] + a \otimes [n^+, n^-]f = n^- \cdot n^+ \cdot a \otimes f + [n^+, n^-] \cdot f. \quad \square$$

The following statement is crusial in the proof of the Theorem.

3.2.5. **Lemma:** Thre exists an isomorphism of left $\mathfrak{n}$-modules $U(\mathfrak{n})^* \cong S^{\mathfrak{n}}_{\mathfrak{n}^-} \otimes_{U(\mathfrak{n}^+)} U(\mathfrak{n}^+)^*$

Proof. Let $f^+ \in U(\mathfrak{n}^+)^*$, $f^- \in U(\mathfrak{n}^-)^*$. Denote the element $(1 \otimes f^-) \otimes f^+ \in S^{\mathfrak{n}}_{\mathfrak{n}^-} \otimes_{U(\mathfrak{n}^+)} U(\mathfrak{n}^+)^*$ by $f^- \otimes f^+$. Then such elements form a base of the vector space $S^{\mathfrak{n}}_{\mathfrak{n}^-} \otimes_{U(\mathfrak{n}^+)} U(\mathfrak{n}^+)^* \cong U(\mathfrak{n}^-)^* \otimes U(\mathfrak{n}^+)^*$. We calculate the action of $\mathfrak{n}$ on $S^{\mathfrak{n}}_{\mathfrak{n}^-} \otimes_{U(\mathfrak{n}^+)} U(\mathfrak{n}^+)^*$ in this base. For $n^- \in \mathfrak{n}^-$ we have

$$n^-(f^- \otimes f^+) = (n^- f^-) \otimes f^+.$$

For $n^+ \in \mathfrak{n}^+$ we have

$$n^+(f^- \otimes f^+) = (n^+ \otimes f^-) \otimes f^+ = (n^+ \cdot (1 \otimes f^-)) \otimes f^+ - [n^+, f^-] \otimes f^+ = f^- \otimes n^+ f^+ - [n^+, f^-] \otimes f^+.$$

Now recall that $\mathfrak{n}^-$ is an ideal in $\mathfrak{n}$ and the actions of the subalgebras $\mathfrak{n}^-$ and $\mathfrak{n}^+$ on $U(\mathfrak{n})^* \cong U(\mathfrak{n}^-)^* \otimes U(\mathfrak{n}^+)^*$ are given by these very formulas. □

We return to the situation of (3.2.1). Note that each module $S_{\mathfrak{n}^k}$ is in fact a $\mathfrak{g}$-bimodule by Lemma 3.1.2.

3.2.6. **Lemma:** The left $\mathfrak{g}$-modules $S_{\mathfrak{n}}$ and $S_{\mathfrak{n}^0} \otimes_{U(\mathfrak{g})} \ldots \otimes_{U(\mathfrak{g})} S_{\mathfrak{n}^{\mathrm{top}-1}}$ are naturally isomorphic to each other.

Proof. We prove by induction by top $-k$ that the $\mathfrak{g}$-modules $S_{F^k\mathfrak{n}}$ and $S_{F^{k+1}\mathfrak{n}} \otimes_{U(\mathfrak{g})} S_{\mathfrak{n}^k}$ are naturally isomorphic to each other. Note that each time the induction hypothesis provides the $\mathfrak{g}$-bimodule structure on $S_{F^{k+1}\mathfrak{n}}$:

$$U(\mathfrak{g}) \hookrightarrow \mathrm{End}_{\mathfrak{g}}(S_{F^{\mathrm{top}-1}\mathfrak{n}}) \overset{\cdot \otimes_{U(\mathfrak{g})} S_{\mathfrak{n}^{\mathrm{top}-2}}}{\longrightarrow} \ldots \overset{\cdot \otimes_{U(\mathfrak{g})} S_{\mathfrak{n}^{k+1}}}{\longrightarrow} \mathrm{End}_{\mathfrak{g}}(S_{F^{k+1}\mathfrak{n}}).$$

Thus by the previous Lemma

$$S_{F^{k+1}\mathfrak{n}} \otimes_{U(\mathfrak{g})} S_{\mathfrak{n}^k} \cong U(\mathfrak{g}) \otimes_{U(F^k\mathfrak{n})} S^{F^k\mathfrak{n}}_{F^{k+1}\mathfrak{n}} \otimes_{U(\mathfrak{n}^k)} U(\mathfrak{n}^k)^* \cong U(\mathfrak{g}) \otimes_{U(F^k\mathfrak{n})} U(F^k\mathfrak{n})^* = S_{F^k\mathfrak{n}}.$$

The Lemma is proved. □

The statement of the Theorem follows immediately from the previous Lemma. □

4. Affine root systems and affine Weyl groups

4.1. **General setting.** In this section we recall some basic facts about affine Weyl groups. In our notations we follow mostly [L1], 1.1 – 1.5. A detailed exposition of the subject can be found in [K], Chapter 6.

4.1.1. Let $(a_{ij})_{i,j\in\{0\}\cup\underline{r}}$ be an irreducible (untwisted) affine Cartan matrix. There are uniquely defined strictly positive integers $d_i, r_i, r_i',\ i \in \{0\} \cup \underline{r}$ such that

(i) $d_i a_{ij} = d_j a_{ji}$ for all i, j and $d_i = 1$ for some i,
(ii) $\sum_i r_i a_{ij} = 0$ for all j and $r_0 = 1$,
(iii) $\sum_j r_j' a_{ij} = 0$ for all i and $r_0' = 1$.

4.1.2. In particular $(a_{ij})_{i,j\in\underline{r}}$ is a Cartan matrix of the finite type.

Let $D = \max_i d_i$, we have $D \in \{1, 2, 3\}$ and for each i, d_i is equal either to 1 or D. We define $\hat{d}_i$ by $d_i\hat{d}_i = D$ for all $i \in \{0\} \cup \underline{r}$.

4.1.3. Let V be a $\mathbf{R}$-vector space with basis $\{h_i | i \in \{0\} \cup \underline{r}\}$ and let V' be the dual vector space. We denote by $\langle \cdot, \cdot \rangle;\ V \times V' \longrightarrow \mathbf{R}$ the canonical bilinear pairing. Let $\{\omega_i | i \in \{0\} \cup \underline{r}\}$ be the basis of V' dual to $\{h_i | i \in \{0\} \cup \underline{r}\}$. Define $\{\alpha_j | j \in \{0\} \cup \underline{r}\}$ by $\langle h_i, \alpha_j \rangle = a_{ij}$. Consider a vector $c := \sum_i r_i h_i \in V$. Then we have $\langle c, \alpha_j \rangle = 0$ for all $j \in \{0\} \cup \underline{r}$ and $\sum_i r_i' \alpha_i = 0$.

4.1.4. For $i \in \{0\} \cup \underline{r}$ we define reflections

$$s_i :\ V \longrightarrow V,\ s_i(y) := y - \langle y, \alpha_i \rangle h_i,\ \text{and}\ s_i :\ V' \longrightarrow V',\ s_i(x) := x - \langle h_i, x \rangle \alpha_i.$$

The subgroup $W \subset GL(V)$ generated by the reflections $s_i,\ i \in \{0\} \cup \underline{r}$, is called the affine Weyl group. We identify W with the subgroup in $GL(V')$ generated by $s_i,\ i \in \{0\} \cup \underline{r}$. The subgroup $\overline{W} \subset W$ generated by the reflections $s_i,\ i \in \underline{r}$, is called the (finite) Weyl group corresponding to the Cartan matrix $(a_{ij})_{i,j\in\underline{r}}$.

4.1.5. The set R (resp. $\overline{R}$) of vectors of V of the form $w(h_i)$ for some $i \in \{0\} \cup \underline{r}$ and $w \in W$ (resp. of the form $w(h_i)$ for some $i \in \underline{r}$ and $w \in \overline{W}$) is called the affine root system (resp. the finite root system). Let R' (resp. $\overline{R}'$) be the set of vectors of V' of the form $\omega(h_i)$ for some $i \in \{0\} \cup \underline{r}$ (resp. for some $i \in \overline{r}$) and some $\omega \in \overline{W}$). The assignment $h_i \mapsto \alpha_i$ extends uniquely to a map

$$h \mapsto h',\ R \longrightarrow R', \text{ such that } \omega(h)' = \omega(h'),\ \omega \in W,\ h \in V.$$

The map restricts to bijection of $\overline{R}$ to $\overline{R}'$.

There is a unique function $h \mapsto d_h$ on R such that it is W-invariant and $d_{h_i} = d_i$ for all $i \in \{0\} \cup \underline{r}$. We define $\hat{d}_h$ by $d_h \hat{d}_h = D$ for all $h \in R$.

4.1.6. **Remark:**

(i) $R = \{\alpha + \hat{d}_\alpha mc | \alpha \in \overline{R}, m \in \mathbf{Z}\}$;
(ii) $R' = \overline{R}'$;
(iii) $(\alpha + \hat{d}_\alpha mc)' = \alpha'$ for all $\alpha \in \overline{R},\ m \in \mathbf{Z}$.

For $h \in R$ we denote by s_h the element of W given by the reflection in V (resp. in V')

$$s_h(y) = y - \langle y, h' \rangle h \text{ (resp. } s_h(x) = x - \langle h, x \rangle h').$$

For any $\alpha \in \overline{R}$ and $m \in \mathbf{Z}$ we set $s_{\alpha,m} = s_h \in W$, where $h = \alpha + \hat{d}_\alpha mc$.

Consider the weight lattice $P \subset V'$ generated by the set $\{\omega_i | i \in \{0\} \cup \underline{r}\}$. Let $Q' \subset V'$ be a free abelian group generated by the set $\{\alpha_i | i \in \underline{r}\}$ and let $Q'' \subset Q'$ be a free abelian group generated by the set $\{\hat{d}_i \alpha_i | i \in \underline{r}\}$. For $z \in Q''$ consider a transvection $\theta_z :\ V' \longrightarrow V'$ given by $\theta_z(x) = x + \langle c, x \rangle z$.

4.1.7. **Lemma:** For $\alpha \in \overline{R}$ and $m \in \mathbf{Z}$ we have $s_{\alpha,0} \circ s_{\alpha,m} = \theta_{\hat{d}_\alpha m \alpha'}$. □

In particular $\theta_z \in W$ for any $z \in Q''$. Consider the map of the sets $\theta :\ Q'' \longrightarrow W,\ z \mapsto \theta_z$, and denote its image by $T \subset W$.

4.1.8. **Lemma:**

(i) The map θ is an injective homomorphism of groups;

(ii) T is a normal subgroup in W;

(iii) W is a semidirect product of T and $\overline{W}$. □

Let V^+ (resp. V^-)be the set of all vectors in V such that all their coefficients with respect to the basis $\{h_i | i \in \{0\} \cup \underline{r}\}$ are nonnegative (resp. nonpositive). Set $R^+ = R \cap V^+,\ R^- = R\ V^-,\ \overline{R} \cap V^+$. We have $R^+ = \{\alpha + \hat{d}_\alpha mc | \alpha \in \overline{R}, m > 0\} \sqcup \overline{R}^+$.

The height of a positive root $\alpha = \sum\limits_{i \in \{0\} \cup \underline{r}} b_i \alpha_i$ is defined as follows: $\mathrm{ht}\, \alpha = \sum\limits_{i \in \{0\} \cup \underline{r}} b_i$.

4.1.9. As usual define the weight $\rho \in V'$ by $\rho(h_i) = 1$ for all $i \in \{0\} \cup \underline{r}$. Consider the *dot* action of W on V':

$$w \cdot \lambda = w(\lambda + \rho) - \rho,\ w \in W,\ \lambda \in V'.$$

Then the dot action of the Weyl group preserves the sets $P_k := \{\lambda \in P | \langle c, \lambda \rangle = k\}$.

From now on we denote the set of dominant weights at the level k by P_k^+:

$$P_k^+ := \{\lambda \in P_k | \langle h_i, \lambda \rangle \geq 0,\ i \in \{0\} \cup \underline{r}\}.$$

4.2. **Semi-infinite Bruhat order on the Weyl group.** Recall that the length of an element of the affine Weyl group w is defined as follows:

$$\ell t(w) := \sharp\{\alpha \in R^+ | w^{-1}(\alpha) \in R^-\}.$$

Remark: The length of $w \in W$ is equal also to the minimal possible length of expression of w via the generators $s_i,\ i \in \{0\} \cup \underline{r}$ (the length of a reduced expression).

4.2.1. For $w \in W$ consider a finite set $R_w := \{\alpha \in R^- | w(\alpha) \in R^+\}$. Let w_1 and w_2 be elements of the Weyl group such that $\ell t(w_1) + \ell t(w_2) = \ell t(w_2 w_1)$. Then $R_{w_2 w_1} = R_{w_2} \sqcup R_{w_1}$. Thus $R_{w_2} \cap -w_1(R_{w_1}) = \emptyset$, where the set $-S \subset R$ consists of elements opposite to the ones of $S \subset R$.

Recall that we have $\sum\limits_{\alpha \in R_w} \alpha = \rho - w^{-1}(\rho)$.

4.2.2. We say that w' follows w in the Weyl group if there exist a reduced expression of w' and $p \in \{1, \dots, \ell t w'\}$ such that

$$w' = s_{i_1} \dots s_{i_{\ell t w'}},\ w = s_{i_1} \dots s_{i_{p-1}} s_{i_{p+1}} \dots s_{i_{\ell t w'}}$$

and $\ell t w = \ell t w' - 1$.

Recall that the usual Bruhat order on the Weyl group is the partial order on W generated by the relation "w' follows w in W". It is denoted by $\geq$.

4.2.3. **Lemma:** The relation

$$\{\text{there exists } \lambda_0 \in Q''^+ \text{ such that for any } \lambda \in Q''^+ \text{ we have } \theta_\lambda \theta_{\lambda_0} w' \geq \theta_\lambda \theta_{\lambda_0} w\}$$

is a partial order on W. □

4.2.4. **Definition:** We call this partial order the semi-infinite Bruhat order on W and denote it by $\geq^{\frac{\infty}{2}}$.

4.2.5. **Remark:** The semi-infinite Bruhat order defined above in fact coincides with the partial order on the affine Weyl group defined in [L2], Section 3, in terms of combinatorics of *alcoves* in $\overline{\mathfrak{h}}^*$. However we will not need the comparison statement. Further details on the partial order can be found in [L2].

4.2.6. Denote the set $\{\alpha \in R^+ | \alpha = \beta + \hat{d}_\beta mc,\ \beta \in \overline{R}^+, m \geq 0\}$ (resp. the set $\{\alpha \in R^+ | \alpha = \beta + \hat{d}_\beta mc,\ \beta \in \overline{R}^-, m > 0\}$) by $R^{\frac{\infty}{2}+}$ (resp. by $R^{\frac{\infty}{2}-}$). Following [FF] we introduce the semi-infinite length function on the affine Weyl group as follows:

$$\ell t^{\frac{\infty}{2}}(w) := \sharp\{\alpha \in R^{\frac{\infty}{2}+} | w(\alpha) \in R^-\} - \sharp\{\alpha \in R^{\frac{\infty}{2}-} | w(\alpha) \in R^-\}.$$

Next we consider the twisted length function on the affine Weyl group. For $u, w \in W$ set

$$\ell t^w(u) := \ell t(w^{-1} u) - \ell t(w^{-1}).$$

In particular for $\mu \in -Q''^+$ and $\nu \in Q''^+$ we have $\ell t^{\theta_\mu}(\theta_\nu) = \ell t(\theta_\nu)$.

4.2.7. **Lemma:** For every $w_1, w_2 \in W$ there exists $\mu_0 \in -Q''^+$ such that for every $\mu \in -Q''^+$ we have

$$\ell t(\theta_{-\mu}\theta_{-\mu_0} w_1) - \ell t(\theta_{-\mu}\theta_{-\mu_0} w_2) = \ell t^{\frac{\infty}{2}}(w_1) - \ell t^{\frac{\infty}{2}}(w_2).$$

In particular for every $\mu \in -Q''^+$ we have $\ell t^{\theta_\mu \theta_{\mu_0}}(w) = \ell t^{\frac{\infty}{2}}(w)$. □

5. Twisted Verma modules over affine Lie algebras and twisting functors

5.1. **Affine Lie algebras.** Fix a Cartan matrix $(a_{ij})_{i,j \in \underline{r}}$ like in 4.1.2 and consider the corresponding semisimple Lie algebra $\overline{\mathfrak{g}}$. In particular $\overline{R}$ is the root system of $\overline{\mathfrak{g}}$. Thus $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}^+ \oplus \overline{\mathfrak{h}} \oplus \overline{\mathfrak{g}}^-$, where $\overline{\mathfrak{h}}$ denotes the Cartan subelgebra in $\overline{\mathfrak{g}}$, and

$$\overline{\mathfrak{g}}^+ = \bigoplus_{\alpha \in \overline{R}^+} \overline{\mathfrak{g}}_\alpha,\ \overline{\mathfrak{g}}^- = \bigoplus_{\alpha \in \overline{R}^-} \overline{\mathfrak{g}}_\alpha.$$

5.1.1. Recall that the affine Lie algebra $\mathfrak{g}$ for the semisimple Lie algebra $\overline{\mathfrak{g}}$ is defined as a central extension of a loop algebra $\mathcal{L}\overline{\mathfrak{g}} := \overline{\mathfrak{g}} \otimes \mathbf{C}\,[t, t^{-1}]$. Namely, $\mathfrak{g} := \mathcal{L}\overline{\mathfrak{g}} \oplus \mathbf{C}K$, and the bracket in $\mathfrak{g}$ is defined as follows:

$$[g_1 \otimes t^n, g_2 \otimes t^m] = [g_1, g_2] \otimes t^{n+m} + \delta_{n+m,0} n B(g_1, g_2) K,$$

where $g_1, g_2 \in \overline{\mathfrak{g}}, n, m \in \mathbf{Z}$, and $B(\cdot,\cdot)$ denotes the Killing form of $\overline{\mathfrak{g}}$.

5.1.2. It is well known that the affine Lie algebra $\mathfrak{g}$ is a Kac-Moody Lie algebra with the Cartan matrix $(a_{ij})_{i,j\in\{0\}\cup\underline{r}}$ and the root system R. Thus $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{h} := \overline{\mathfrak{h}} \oplus \mathbf{C}K = V \otimes_{\mathbf{R}} \mathbf{C}$ and

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha = \overline{\mathfrak{g}}^+ \oplus \overline{\mathfrak{g}} \otimes t\mathbf{C}[t],\ \mathfrak{n}^- = \bigoplus_{\alpha\in R^-} \mathfrak{g}_\alpha = \overline{\mathfrak{g}}^- \oplus \overline{\mathfrak{g}} \otimes t^{-1}\mathbf{C}[t^{-1}].$$

5.1.3. Recall the definition of the Chevalley generators for $\mathfrak{g}$. Set $e_{\alpha_i} := e_i,\ e_{-\alpha_i} := f_i,\ i = 1,\ldots,r$, where e_i, f_i are the Chevalley generators for $\overline{\mathfrak{g}}$, $[e_i, f_i] = h_i$. Let α_{top} be the highest root of the root system $\overline{R}$. Choose a vector h_{top} in $[\overline{\mathfrak{g}}_{\alpha_{\text{top}}}, \overline{\mathfrak{g}}_{-\alpha_{\text{top}}}] \subset \overline{\mathfrak{h}}$ such that $\alpha_{\text{top}}(h_{\text{top}}) = 2$. Fix $e_{\text{top}} \in \overline{\mathfrak{g}}_{\alpha_{\text{top}}}$ and $f_{\text{top}} \in \overline{\mathfrak{g}}_{-\alpha_{\text{top}}}$ such that $[e_{\text{top}}, f_{\text{top}}] = h_{\text{top}}$. Then set

$$e_{\alpha_0} := f_{\text{top}} \otimes t,\ e_{-\alpha_0} := e_{\text{top}} \otimes t^{-1}.$$

5.1.4. We introduce a grading on $\mathfrak{g}$ putting $\deg e_i = 1$, $\deg f_i = -1$ and $\deg h_i = 0$ for all $i \in \{0\} \cup \underline{r}$.

5.2. **Categories of $\mathfrak{g}$-modules.** Consider the category $\mathcal{M}$ of $\mathfrak{h}^* \times \mathbf{Z}$-graded $\mathfrak{g}$-modules $M = \bigoplus\limits_{\lambda\in\mathfrak{h}^*, t\in\mathbf{Z}} M_{\lambda,t}$ such that

(i) for every $i \in \{0\} \cup \underline{r}$ the Chevalley generators $e_i :\ M_{\lambda,t} \longrightarrow M_{\lambda+\alpha_i,t+1},\ f_i :\ M_{\lambda,t} \longrightarrow M_{\lambda-\alpha_i,t-1}$;
(ii) every $h \in \mathfrak{h}$ acts on $M_{\lambda,t}$ by scalar $\lambda(h) = \langle h, \lambda\rangle$.

Morphisms in $\mathcal{M}$ are morphisms of $\mathfrak{g}$-modules that preserve $\mathfrak{h}^* \times \mathbf{Z}$-gradings.

Fix a nonnegative integer $k \in \mathbf{Z}_{\geq 0}$. Denote by $U_k(\mathfrak{g})$ the quotient algebra of the universal enveloping algebra of $\mathfrak{g}$ by the relation $K - k = 0$. Consider a full subcategory $\mathcal{M}_k$ in $\mathcal{M}$ of modules M such that K acts by the scalar k on M. Denote by $\operatorname{supp} M$ the set $\{\lambda \in \mathfrak{h}^* | M_{\lambda,t} \neq 0 \text{ for some } t \in \mathbf{Z}\}$. Then for every $M \in \mathcal{M}_k$ the set $\operatorname{supp} M$ is contained in the set $\mathfrak{h}^*_k := \{\lambda \in \mathfrak{h}^* | \langle c, \lambda\rangle = k\}$. The set $\mathfrak{h}^*_k$ is preserved both by the linear action of the affine Weyl group on $\mathfrak{h}^*$ and by the dot action.

5.2.1. Let D be the contragradient duality functor:

$$D :\ \mathcal{M} \longrightarrow \mathcal{M},\ D(M) := \bigoplus_{\lambda\in\mathfrak{h}^*, t\in\mathbf{Z}} M^*_{-\lambda,-t},\ M \in \mathcal{M}.$$

The left action of $\mathfrak{g}$ on $D(M)$ is provided by the composition of the usual right action of $\mathfrak{g}$ on the dual module with the antipode and the Chevalley involution of $\mathfrak{g}$. In particular D preserves $\mathcal{M}_k$.

5.2.2. We define the character of a $\mathfrak{g}$-module $M \in \mathcal{M}$ such that $\dim M_{\lambda,t} < \infty$ for all $\lambda \in \mathfrak{h}^*, t \in \mathbf{Z}$ by

$$\operatorname{ch} M := \sum_{\lambda\in\mathfrak{h}^*, t\in\mathbf{Z}} \dim M_{\lambda,t} e^\lambda q^t.$$

Here q is a formal variable and e^λ is a formal expression.

5.2.3. As usual let $\mathcal{O}$ denote the category of $\mathfrak{h}^*$-graded $\mathfrak{g}$-modules $M = \bigoplus\limits_{\lambda\in\mathfrak{h}^*} M_\lambda$ such that

(i) for every $i \in \{0\} \cup \underline{r}$ the Chevalley generators $e_i :\ M_\lambda \longrightarrow M_{\lambda+\alpha_i,},\ f_i :\ M_\lambda \longrightarrow M_{\lambda-\alpha_i,}$;
(ii) every $h \in \mathfrak{h}$ acts on M_λ by scalar $\lambda(h) = \langle h, \lambda\rangle$;
(iii) $\dim M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$;
(iv) there exist $\lambda_1, \ldots, \lambda_n \in \mathfrak{h}^*$ such that

$$M_\mu \neq 0 \text{ only when } \mu \in \lambda_1 + R^- \cup \ldots \cup \lambda_n + R^-.$$

5.2.4. We recall the definition of the Casimir endomorphism $\Gamma_M :\ M \longrightarrow M$ for $M \in \mathcal{O}$. Choose a homogenious base $\{e_\alpha | \alpha \in R\} \cup \{e_{0,i} | i \in \{0\} \cup \underline{r}\}$ in $\mathfrak{g}$ and let $\{e^\alpha | \alpha \in R\} \cup \{e_0^i | i \in \{0\} \cup \underline{r}\}$ be the dual base with respect to the Killing form on $\mathfrak{g}$. For $M \in \mathcal{O}$ we define the Casimir operator by

$$\Gamma_M|_{M_\nu} = (\nu + \rho, \nu + \rho)\operatorname{Id}_{M_\nu} + 2\sum_{\alpha\in R^+} e^\alpha e_\alpha$$

where $e^\alpha e_\alpha \in U(\mathfrak{g})$. It is known that Γ_M commutes with the action of $\mathfrak{g}$. For $M \in \mathcal{O}$ and $\kappa \in \mathbf{C}$ set

$$M_\kappa := \{m \in M | \text{ there exists } n \in \mathbf{N} \text{ such that } (\Gamma_M - \kappa)^n m = 0\}$$

5.2.5. **Lemma:** M_κ is a submodule in M and $M = \bigoplus_{\kappa\in\mathbf{C}} M_\kappa$. □

5.3. **Affine BGG resolution.** Let $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) be the positive (resp. negative) Borel subalgebra in $\mathfrak{g}$, $\mathfrak{b}^+ := \mathfrak{h} \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ (resp. $\mathfrak{b}^- := \mathfrak{h} \oplus \bigoplus_{\alpha\in R^-} \mathfrak{g}_\alpha$). Then $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}^+$ and $\mathfrak{b}^+/\mathfrak{n}^+ = \mathfrak{h}$. For the one dimensional $\mathfrak{h}$-module $\mathbf{C}(\lambda)$ the Verma module $U_k(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbf{C}(\lambda)$ is denoted by $M(\lambda)$. Evidently $M(\lambda)$ is free as a $U(\mathfrak{n}^-)$-module and belongs to $\mathcal{O}$ and to $\mathcal{M}_k$ if we put the $\mathbf{Z}$-grading of the highest weight vector $v_\lambda \in M(\lambda)$ equal to zero.

Recall the construction of the BGG resolution in the case of affine Lie algebras (see e. g [RW]). For a fixed positive integral level k choose a dominant weight $\lambda \in P_k^+$. Consider the unique simple quotient module of $M(\lambda)$ denoted by $L(\lambda)$. Then there exists a left resolution of $L(\lambda)$ that consists of direct sums of Verma modules. Namely recall first the construction of the standard resolution of the trivial $\mathfrak{g}$-module $\underline{\mathbf{C}}$ relative to $\mathfrak{b}^+$.

5.3.1. *Standard resolutions.* Consider the standard complex $U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}) \otimes \underline{\mathbf{C}}$ for the computation of the Lie algebra homology of $\mathfrak{g}$ with coefficients in $U(\mathfrak{g})$. Clearly

$$H^{\neq 0}(U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}) \otimes \underline{\mathbf{C}}) = 0,\ H^0(U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}) \otimes \underline{\mathbf{C}}) = \underline{\mathbf{C}}$$

and $U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}) \cong U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{b}) \otimes \Lambda^\bullet(\mathfrak{g}/\mathfrak{b})$ as a vector space. We introduce a filtration of the complex as follows:

$$F^m(U(\mathfrak{g}) \otimes \Lambda^n(\mathfrak{g})) := \bigoplus_{\substack{p+q=n\\ q\leq m}} U(\mathfrak{g}) \otimes \Lambda^p(\mathfrak{b}) \otimes \Lambda^q(\mathfrak{g}/\mathfrak{b}).$$

Clearly the differential in the complex preserves the filtration. Consider the corresponding spectral sequence. We have

$$E_1^{p,q} = H_{-p}(\mathfrak{b}, U(\mathfrak{g}) \otimes \Lambda^{-q}(\mathfrak{g}/\mathfrak{b})) = \delta_{p,0} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \Lambda^{-q}(\mathfrak{g}/b).$$

Thus the spectral sequence converges in E_2. On the other hand $E_\infty^{p,q} = \delta_{p,0}\delta_{q,0}\underline{\mathbf{C}}$ and we obtain the required standard resolution of the trivial $\mathfrak{g}$-module $\underline{\mathbf{C}}$ with relative to $\mathfrak{b}$:

$$D^\bullet := U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \Lambda^\bullet(\mathfrak{g}/\mathfrak{b}^+),\ d:\ D^{-k} \longrightarrow D^{-k+1},$$

$$d(u \otimes \overline{a}_1 \wedge \ldots \wedge \overline{a}_k) = \sum_{i=1}^{k} (-1)^i u a_i \otimes \overline{a}_1 \wedge \ldots \wedge \overline{a}_{i-1} \wedge \overline{a}_{i+1} \wedge \ldots \wedge \overline{a}_k$$

$$+ \sum_{i<j} (-1)^{i+j} u \otimes \overline{[a_i, a_j]} \wedge \overline{a}_1 \wedge \ldots \wedge \overline{a}_{i-1} \wedge \overline{a}_{i+1} \wedge \ldots \wedge \overline{a}_{j-1} \wedge \overline{a}_{i+1} \wedge \ldots \wedge \overline{a}_k,$$

where $\overline{a}$ denotes the image of a in $\mathfrak{g}/\mathfrak{b}^+$.

5.3.2. **Remark:** Suppose we have two subalgebras $\mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \mathfrak{g}$. Then the identity map $U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g})$ is a morphism of filtered complexes: $F^m_{\mathfrak{b}_1}(U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g})) \subset F^m_{\mathfrak{b}_2}(U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}))$. Thus we have a canonical morphism of the standard resolutions $D^\bullet_{\mathfrak{b}_1} \longrightarrow D^\bullet_{\mathfrak{b}_2}$.

5.3.3. Consider the standard resolution of the simple module $L(\lambda)$ relative to $\mathfrak{b}^+$: $D^\bullet(L(\lambda)) := D^\bullet \otimes L(\lambda)$ with the structure of $\mathfrak{g}$-module using comultiplication on $U(\mathfrak{g})$. Then $D^\bullet(L(\lambda)) = \bigoplus_{\kappa\in\mathbf{C}} (D^\bullet(L(\lambda)))_\kappa$.

Taking the direct summand with $\kappa = (\lambda+\rho, \lambda+\rho)$ we obtain a resolution of $L(\lambda)$ filtered by Verma modules. The folowing statement says that the filtration splits.

5.3.4. **Theorem:** (see [RW] 9.7) There exists a resolution $B^\bullet(\lambda) = (D^\bullet(L(\lambda)))_{(\lambda+\rho,\lambda+\rho)}$ of $L(\lambda)$ in $\mathcal{O}$ and in $\mathcal{M}_k$ of the form

$$\ldots \longrightarrow \bigoplus_{\substack{w\in W,\\ \ell t(w)=m}} M(w\cdot\lambda)\langle -\operatorname{ht}(\lambda - w\cdot\lambda)\rangle \longrightarrow \ldots \longrightarrow \bigoplus_{\substack{w\in W,\\ \ell t(w)=1}} M(w\cdot\lambda)\langle -\operatorname{ht}(\lambda - w\cdot\lambda)\rangle \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

Here as usual $\langle\cdot\rangle$ denotes the shift of the $\mathbf{Z}$-grading in $\mathcal{M}$. □

5.4. **Twisting functors.** Recall that the affine Weyl group has a geometric description in terms of affine Lie groups. Let $\hat{\mathcal{L}G}$ be the central extension of the loop group of $\overline{\mathfrak{g}}$. Let $T \subset \hat{\mathcal{L}G}$ be the Cartan subgroup. Denote by $N(T)$ its normalizer in $\hat{\mathcal{L}G}$.

5.4.1. **Lemma:** The affine Weyl group W is isomorphic to $N(T)/T$. $\square$

For every $w \in W$ fix its representative $\dot{w} \in N(T)$. The normalizer of T acts on $\mathfrak{g}$ by adjunction and this action commutes with the bracket in $\mathfrak{g}$ and shifts the weight decomposition. In particular we have maps

$$\dot{w} : \mathfrak{g} \longrightarrow \mathfrak{g},\ \mathfrak{g}_\alpha \longrightarrow \mathfrak{g}_{w(\alpha)}.$$

5.4.2. We introduce the functors of the twist of the $\mathfrak{h}^*$-grading. We define $T_w :\ \mathcal{M}_k \longrightarrow \mathcal{M}_k$ as follows. For $M \in \mathcal{M}_k$ set

$$T_w(M) = \bigoplus_{\lambda\in\mathfrak{h}^*, t\in\mathbf{Z}} T_w(M)_{\lambda,t},\ T_w(M)_{\lambda,t} := M_{w(\lambda),t+\mathrm{ht}(w(\lambda)-\lambda)},\ e_\alpha \in \mathfrak{g}_\alpha,\ m_\lambda \in T_w(M)_{\lambda,t} \text{ then}$$

$$e_\alpha \cdot m_\lambda := \dot{w}(e_\alpha)(m_\lambda) \in M_{w(\lambda+\alpha),t+\mathrm{ht}(w(\lambda+\alpha)-\lambda)} = M_{\lambda+\alpha,t+\mathrm{ht}(w(\lambda+\alpha)-\lambda-\alpha)+\mathrm{ht}\,\alpha} = T_w(M)_{\lambda+\alpha,t+\mathrm{ht}\,\alpha}.$$

Evidently T_w is an equivalence of categories and the opposite functor is $T_{w^{-1}}$.

5.4.3. Fix $w \in W$. Let $\mathfrak{n}_w$ be a finite dimensional nilpotent subalgebra in $\mathfrak{g}$:

$$\mathfrak{n}_w := \mathfrak{n}^- \cap \dot{w}^{-1}(\mathfrak{n}^+) = \bigoplus_{\alpha\in R_w} \mathfrak{g}_\alpha.$$

Consider the semiregular $\mathfrak{g}$-module with respect to $\mathfrak{n}_w$ at the level k:

$$S_{\mathfrak{n}_w} = \mathrm{Ind}_{U(\mathfrak{n}_w)}^{U_k(\mathfrak{g})} \mathrm{Coind}_{\mathbf{C}}^{U(\mathfrak{n}_w)} \underline{\mathbf{C}}.$$

Then by 3.2.1 there exists an inclusion of algebras $\sigma_w :\ U_k(\mathfrak{g}) \hookrightarrow \mathrm{End}_{\mathfrak{g}}(S_{\mathfrak{n}_w})$.

Like in the third section, for $\alpha \in R_w$ consider the $\mathfrak{g}$-module $S_{\mathfrak{n}_\alpha} := U_k(\mathfrak{g}) \otimes_{U(\mathfrak{n}_\alpha)} U(\mathfrak{n}_\alpha)^*$. By 3.2.1 there also exists an inclusion of algebras $\sigma_\alpha :\ U_k(\mathfrak{g}) \hookrightarrow \mathrm{End}_{\mathfrak{g}}(S_{\mathfrak{n}_\alpha})$ (see 3.1.1). As before identify $U(\mathfrak{n}_\alpha)^*$ with $\mathbf{C}[x_\alpha]$ so that $e_\alpha \in \mathfrak{n}_\alpha$ acts by $\partial/\partial x_\alpha$.

5.4.4. **Lemma:** For every $h \in \mathfrak{h}$ we have $\sigma_\alpha(h) = h - \alpha(h) e_\alpha \otimes x_\alpha$.

Proof. Follows immediately from 3.1.2. $\square$

5.4.5. Consider a functor

$$S_w :\ \mathcal{M}_k \longrightarrow U_k(\mathfrak{g})\text{-mod},\ S_w(M) := S_{\mathfrak{n}_w} \otimes_{U_k(\mathfrak{g})} M.$$

We will show that in fact S_w can be dafined as a functor from $\mathcal{M}_k$ to $\mathcal{M}_k$.

Remark: Note that $S_w(M) \cong U(\mathfrak{n}_w)^* \otimes_{U(\mathfrak{n}_w)} M$ as a vector space (and even as a $U(\mathfrak{n}_w)$-module).

Let $f \in U(\mathfrak{n}_w)^*_{-\lambda} = (U(\mathfrak{n}_w)_\lambda)^*$ and $m \in M_{\mu,t}$. Then set the $\mathfrak{h}^* \times \mathbf{Z}$-grading of $f \otimes m \in S_w(M)$ equal to $(\mu - \lambda + \rho - w^{-1}(\rho), t + \mathrm{ht}(\rho - w^{-1}(\rho)))$. The grading is well defined with respect to the action of $\mathfrak{g}$. It remains to check that $S_w(M)$ is $\mathfrak{h}$-semisimple. But this follows from 5.4.4 and 3.2.6. We have proved the following statement.

5.4.6. **Lemma:** S_w defines a functor $\mathcal{M}_k \longrightarrow \mathcal{M}_k$. $\square$

5.4.7. **Definition:** For $w \in W$ the functor of twist by w

$$\Phi_w := T_w \circ S_w,\ \Phi_w :\ \mathcal{M}_k \longrightarrow \mathcal{M}_k.$$

Let us describe the image of a Verma module under Φ_w.

5.4.8. **Lemma:** $\mathrm{ch}\,\Phi_w(M(\lambda)) = \mathrm{ch}\,M(w\cdot\lambda)\langle -\mathrm{ht}(\lambda - w\cdot\lambda)\rangle$.

Proof. Follows immediately from the definition of the $\mathfrak{h}^* \times \mathbf{Z}$-grading on $T_w \circ S_w(M(\lambda))$. $\square$

In particular the highest weight vector $1 \otimes v_\lambda \in \Phi_w(M(\lambda))$ has the weight $w \cdot \lambda$.

5.4.9. **Definition:** We call the $\mathfrak{g}$-module $M_w(w\cdot\lambda) := \Phi_w(M(\lambda))\langle \mathrm{ht}(\lambda - w\cdot\lambda)\rangle$ the *twisted Verma module* of the weight $w \cdot \lambda$.

5.4.10. **Remark:** Consider the nilpotent subalgebras $\dot{w}(\mathfrak{n}_w)=\dot{w}(\mathfrak{n}^-)\cap\mathfrak{n}^+\subset\dot{w}(\mathfrak{n}^-)\subset\mathfrak{g}$. Then when restricted to $\dot{w}(\mathfrak{n}^-)$ the twisted Verma module $M_w(w\cdot\lambda)$ is isomorphic to the $\dot{w}(\mathfrak{n}^-)$-module $S_{\dot{w}(\mathfrak{n}_w)}^{\dot{w}(\mathfrak{n}^-)}$.

5.5. **Twisted BGG resolutions.** Fix $w\in W$. Consider the complex in $\mathcal{M}_k$ of the form $\Phi_w(B^\bullet(\lambda))$. Using the fact that $B^\bullet(\lambda)$ is a $U(\mathfrak{n}_w)$-free resolution of $L(\lambda)$ we see that up to a $\mathbf{Z}$-grading shift

$$H^{-\bullet}(\Phi_w(B^\bullet(\lambda)))=\mathrm{Tor}_\bullet^{\mathfrak{n}_w}(U(\mathfrak{n}_w)^*,L(\lambda)).$$

as vector spaces.

5.5.1. **Proposition:** $H^{\neq -l(w)}(\Phi_w(B^\bullet(\lambda)))=0,\ H^{-l(w)}(\Phi_w(B^\bullet(\lambda)))=L(\lambda)$ as a $\mathfrak{g}$-module.

5.6. **Proof of the Proposition 5.5.1.** We present first several Lemmas. Let $\mathfrak{n}$ be a negatively graded finite dimensional Lie algebra.

5.6.1. **Lemma:** $\mathrm{Tor}^{\mathfrak{n}}_{\neq\dim\mathfrak{n}}(U(\mathfrak{n})^*,\underline{\mathbf{C}})=0,\ \mathrm{Tor}^{\mathfrak{n}}_{\dim\mathfrak{n}}(U(\mathfrak{n})^*,\underline{\mathbf{C}})=\mathbf{C}$.

Proof. Consider the standard complex for the computation of $\mathfrak{n}$-homology $K^\bullet(U(\mathfrak{n})^*):=U(\mathfrak{n})^*\otimes\Lambda^\bullet(\mathfrak{n})\otimes\underline{\mathbf{C}}$. Then the PBW filtration on $U(\mathfrak{n})$ induces filtrations both on $U(\mathfrak{n})^*$ and on $\Lambda(\mathfrak{n})$ (such that for any base vector e_i of $\mathfrak{n}\subset\Lambda(\mathfrak{n})$ we have $e_i\in F^1\Lambda(\mathfrak{n})$ and for any dual base vector $e_i^*\in U(\mathfrak{n})^*$ we have $e_i^*\in F^{-1}U(\mathfrak{n})^*$). The differential in $K^\bullet(U(\mathfrak{n})^*)$ preserves the filtration. The complex $\mathrm{gr}^F K^\bullet(U(\mathfrak{n})^*)$ is isomorphic to the coKoszul complex $S(\mathfrak{n})^*\otimes\Lambda^\bullet(\mathfrak{n})$ of the algebra $\Lambda(\mathfrak{n})$. Thus

$$H^{\neq-\dim\mathfrak{n}}(\mathrm{gr}^F K^\bullet(U(\mathfrak{n})^*))=0,H^{-\dim\mathfrak{n}}(\mathrm{gr}^F K^\bullet(U(\mathfrak{n})^*))=\mathbf{C}.$$

Now the grading on $\mathfrak{n}$ induces a grading on $K^\bullet(U(\mathfrak{n})^*)$. The spectral sequence of the filtration F is finite in each grading component of the complex $K^\bullet(U(\mathfrak{n})^*)$. Thus it converges. $\square$

5.6.2. **Corollary:** Let M be a finite dimensional graded $\mathfrak{n}$-module. Then

$$\mathrm{Tor}^{\mathfrak{n}}_{\neq\dim\mathfrak{n}}(U(\mathfrak{n})^*,M)=0,\ \mathrm{Tor}^{\mathfrak{n}}_{\dim\mathfrak{n}}(U(\mathfrak{n})^*,M)=M$$

as a graded vector space. $\square$

5.6.3. **Lemma:** Let M be a graded $\mathfrak{n}$-module with a filtration

$$F^0M\subset\ldots\subset F^mM\subset\ldots\subset M,\ \dim F^mM<\infty \text{ for every } m\in\mathbf{Z}_{>0}.$$

Then $\mathrm{Tor}^{\mathfrak{n}}_{\neq\dim\mathfrak{n}}(U(\mathfrak{n})^*,M)=0,\ \mathrm{Tor}^{\mathfrak{n}}_{\dim\mathfrak{n}}(U(\mathfrak{n})^*,M)=M$.

Proof. Recall that the functor $\varinjlim$ is exact. The standard complex for the computation of $\mathfrak{n}$-homology

$$U(\mathfrak{n})^*\otimes\Lambda(\mathfrak{n})\otimes M=\varinjlim_m U(\mathfrak{n})^*\otimes\Lambda(\mathfrak{n})\otimes F^mM.$$

Thus we have

$$H^\bullet(U(\mathfrak{n})^*\otimes\Lambda(\mathfrak{n})\otimes M)=\varinjlim_m H^\bullet(U(\mathfrak{n})^*\otimes\Lambda(\mathfrak{n})\otimes F^mM)=\delta_{\bullet,-\dim\mathfrak{n}}\varinjlim_m F^mM=\delta_{\bullet,-\dim\mathfrak{n}}M.\quad\square$$

Now recall that $L(\lambda)$ is an integrable $\mathfrak{g}$-module, in particular it is a union of finite dimensional $\mathfrak{n}_w$-modules. Using the previous Lemma we see that the statement of the Proposition is proved on the level of $\mathfrak{h}^*\times\mathbf{Z}$-graded vector spaces. But an integrable $\mathfrak{g}$-module is completely determined by its character. $\square$

5.6.4. **Corollary:** There exists a complex $B_w^\bullet(\lambda)$ in $\mathcal{M}_k$ of the form

$$\ldots\longrightarrow\bigoplus_{\substack{v\in W,\\ \ell t(v)=m}} M_w(wv\cdot\lambda)\langle-\mathrm{ht}(\lambda-wv\cdot\lambda)\rangle\longrightarrow\ldots$$

$$\longrightarrow\bigoplus_{\substack{v\in W,\\ \ell t(v)=1}} M_w(wv\cdot\lambda)\langle-\mathrm{ht}(\lambda-wv\cdot\lambda)\rangle\longrightarrow M_w(w\cdot\lambda)\langle-\mathrm{ht}(\lambda-w\cdot\lambda)\rangle\longrightarrow 0$$

such that $H^{\neq-\ell t(w)}(B_w^\bullet(\lambda))=0,\ H^{-\ell t(w)}(B_w^\bullet(\lambda))=L(\lambda)$. Here as usual $\langle\cdot\rangle$ denotes the shift of the $\mathbf{Z}$-grading. $\square$

5.6.5. Recall that in 4.2.6 we have defined the twisted length function on the affine Weyl group with the twist $w \in W$ by

$$\ell t^w(u) = \ell t(w^{-1}u) - \ell t(w^{-1}).$$

5.6.6. **Corollary:** The complex $B_w(\lambda)[-\ell t(w)]$ can be rewritten as follows:

$$\ldots \longrightarrow \bigoplus_{\substack{v\in W,\\ \ell t^w(v)=m}} M_w(v\cdot\lambda)\langle -\operatorname{ht}(\lambda - v\cdot\lambda)\rangle \longrightarrow \ldots \longrightarrow \bigoplus_{\substack{v\in W,\\ \ell t^w(v)=0}} M_w(v\cdot\lambda)\langle -\operatorname{ht}(\lambda - v\cdot\lambda)\rangle \longrightarrow \ldots$$

$$\longrightarrow \bigoplus_{\substack{v\in W,\\ \ell t^w(v)=-\ell t(w)+1}} M_w(v\cdot\lambda)\langle -\operatorname{ht}(\lambda - v\cdot\lambda)\rangle \longrightarrow M_w(w\cdot\lambda)\langle -\operatorname{ht}(\lambda - w\cdot\lambda)\rangle \longrightarrow 0. \quad \square$$

5.6.7. **Definition:** We call the complex $B_w^\bullet(\lambda)$ the *twisted BGG resolution* of the module $L(\lambda)$ with the twist w.

6. Limit procedure and semi-infinite BGG resolution

6.1. **Recollections of semi-infinite Lie algebra cohomology.** Here we present a list of the definitions and facts about semi-infinite cohomology of Lie algebras to be used later.

6.1.1. *Critical cocycle.* Let $V = \bigoplus_{n\in\mathbf{Z}} V_n$ be a graded vector space such that $\dim V_n < \infty$ for $n > 0$. Denote the space $\bigoplus_{n\le 0} V_n$ (resp. the space $\bigoplus_{n>0} V_n$) by V^- (resp. by V^+). Consider the Lie algebra $\mathfrak{gl}(V)$ of linear transformations of V satisfying the condition:

$$a \in \mathfrak{gl}(V),\ a = (a_{ij}), \Longrightarrow \exists m \in \mathbf{N}:\ a_{ij} = 0 \text{ for } |i-j| > m.$$

$\mathfrak{gl}(V)$ has a well known central extension given by the 2-cocycle ω_0:

$$\omega_0(a_1, a_2) = \operatorname{tr}(\pi_{V^+} \circ [a_1, a_2] - [\pi_{V^+} \circ a_1, \pi_{V^+} \circ a_2]).$$

Here π_{V^+} denotes the projection $V \longrightarrow V_+$ with the kernel V_-. Note that the trace is well defined on maps $V_+ \longrightarrow V_+$ nontrivial on a finite number of the grading components.

6.1.2. Let $\mathfrak{a}$ be a graded Lie algebra such that all $\mathfrak{a}_m$ are finite dimensional. Then $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ as a vector space, where

$$\mathfrak{a}^+ := \bigoplus_{m>0} \mathfrak{a}_m,\ \mathfrak{a}^- := \bigoplus_{m\le 0} \mathfrak{a}_m.$$

For any graded $\mathfrak{a}$-module V we have a morphism of Lie algebras $\mathfrak{a} \longrightarrow \mathfrak{gl}(V)$. The inverse image of the 2-cocycle ω_0 is denoted by ω_0^V. In particular the adjoint representation of $\mathfrak{a}$ provides the 2-cocycle $\omega_0^{\mathfrak{a}}$ called the *critical cocycle* of $\mathfrak{a}$.

6.1.3. *Feigin standard complex.* Choose a homogenious base $\{e_i | i \in \mathbf{Z}\}$ in $\mathfrak{a}$ and let $\{c_{i,j}^k\}_{i,j,k\in\mathbf{Z}}$ denote the structure constants of $\mathfrak{a}$ in this base. For a graded $\mathfrak{a}$-module $M = \bigoplus_{m\le m_0} M_m$ consider a graded vector space

$$C^{\frac{\infty}{2}+\bullet}(M) := \Lambda^\bullet(\mathfrak{a}^{+*}) \otimes \Lambda^\bullet(\mathfrak{a}^-) \otimes M,\ \deg \overline{e}_n^* = 1,\ \deg \overline{e}_m = -1,\ \overline{e}_m \in \mathfrak{a}^-,\ \overline{e}_n^* \in \mathfrak{a}^+.$$

Then the element $D := \sum_{i\in\mathbf{Z}} \overline{e}_i^* \otimes e_i + \sum_{i<j,k} c_{i,j}^k : \overline{e}_i^* \overline{e}_j^* \overline{e}_k :$ can be considered as an endomorphism of $C^{\frac{\infty}{2}+\bullet}(M)$. Here as usual $:\ :$ denotes the normal ordering. It is known that the cohomology class of the critical 2-cocycle $\omega_0^{\mathfrak{a}}$ of the Lie algebra $\mathfrak{a}$ is the only obstruction for $D^2 = 0$. We suppose below that the cohomology class of $\omega_0^{\mathfrak{a}}$ equals to zero and $D^2 = 0$. Then $C^{\frac{\infty}{2}+\bullet}(M)$ becomes a complex of vector spaces with the differential D called the *Feigin standard complex.*

6.1.4. **Remark:** Assume that $\mathfrak{a}$ has a decreasing filtration by subalgebras

$$\mathfrak{a} = F^0\mathfrak{a} \supset \ldots \supset F^m\mathfrak{a} \supset \ldots,\ [F^m\mathfrak{a}, F^n\mathfrak{a}] \subset F^{m+n}\mathfrak{a}$$

and all the inclusions $F^{m+1}\mathfrak{a} \subset F^m\mathfrak{a}$ are strict. In this case the 2-cocycle ω_0 itself equals to zero.

6.1.5. **Definition:** The cohomologies of the complex $C^{\frac{\infty}{2}+\bullet}(M)$ are called *semi-infinite cohomology spaces* of the Lie algebra $\mathfrak{a}$ with coefficients in the module M:

$$H^{\frac{\infty}{2}+\bullet}(M) := H^\bullet(C^{\frac{\infty}{2}+\bullet}(M)).$$

6.1.6. Let $\mathfrak{a} = \bigoplus_{m\in\mathbf{Z}} \mathfrak{a}_m$ be a graded Lie algebra containing two graded Lie subalgebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$ such that $\mathfrak{a}_{>0} = \mathfrak{b}_{1>0}$ and $\mathfrak{a}_{\leq 0} = \mathfrak{b}_{2\leq 0}$. We suppose that the condition 6.1.4 is satisfied and the obstruction 2-cocycle is zero for all three Lie algebras.

6.1.7. **Lemma:** For a graded $\mathfrak{a}$-module $M = \bigoplus_{m<m_0} M_m$ there exist natural morphisms

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, M) \longrightarrow H^{\frac{\infty}{2}+\bullet}(\mathfrak{b}_2, M) \text{ and } H^{\frac{\infty}{2}+\bullet}(\mathfrak{b}_1, M) \longrightarrow H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, M).$$

Proof. There exist natural morphisms of the graded vector spaces

$$C^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, M) \longrightarrow C^{\frac{\infty}{2}+\bullet}(\mathfrak{b}_2, M) \text{ and } C^{\frac{\infty}{2}+\bullet}(\mathfrak{b}_1, M) \longrightarrow C^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, M).$$

One has to check that the maps commute with the differentials and provide morphisms of the standard complexes for the computation of Lie algebra semi-infinite cohomology. □

6.1.8. **Remark:** The statement of the previous Lemma follows easily (and perhaps more naturally) from the general homological algebra (see [V] 3.9, [Ar1] Appendix B). Namely semi-infinite cohomology of a Lie algebra $\mathfrak{a}$ satisfying 6.1.4 can be viewed as an exotic derived functor of a functor $M \mapsto \mathrm{Hom}_{\mathfrak{a}}(\underline{\mathbf{C}}, S^{\mathfrak{a}}_{\mathfrak{a}_{\leq 0}} \otimes_{U(\mathfrak{a})} M)$. Now in the Lemma we have

$$S^{\mathfrak{a}}_{\mathfrak{a}_{\leq 0}} = \mathrm{Ind}^{\mathfrak{a}}_{\mathfrak{b}_2} S^{\mathfrak{b}_2}_{\mathfrak{b}_{2\leq 0}}, \text{ and } S^{\mathfrak{a}}_{\mathfrak{a}_{\leq 0}} = \mathrm{Coind}^{\mathfrak{a}}_{\mathfrak{b}_1} S^{\mathfrak{b}_1}_{\mathfrak{b}_{1\leq 0}}$$

Thus there exist natural morphisms

$$\mathrm{Hom}_{\mathfrak{a}}(\underline{\mathbf{C}}, S^{\mathfrak{a}}_{\mathfrak{a}_{\leq 0}} \otimes_{U(\mathfrak{a})} M) \longrightarrow \mathrm{Hom}_{\mathfrak{b}_2}(\underline{\mathbf{C}}, S^{\mathfrak{a}}_{\mathfrak{a}_{\leq 0}} \otimes_{U(\mathfrak{a})} M) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{b}_2}(\underline{\mathbf{C}}, S^{\mathfrak{b}_2}_{\mathfrak{b}_{2\leq 0}} \otimes_{U(\mathfrak{b}_2)} M),$$
$$\mathrm{Hom}_{\mathfrak{b}_1}(\underline{\mathbf{C}}, S^{\mathfrak{b}_1}_{\mathfrak{b}_{1\leq 0}} \otimes_{U(\mathfrak{b}_1)} M) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{a}}(\underline{\mathbf{C}}, S^{\mathfrak{a}}_{\mathfrak{a}_{\leq 0}} \otimes_{U(\mathfrak{b}_1)} M) \longrightarrow \mathrm{Hom}_{\mathfrak{a}}(\underline{\mathbf{C}}, S^{\mathfrak{a}}_{\mathfrak{a}_{\leq 0}} \otimes_{U(\mathfrak{a})} M)$$

that provide the required morphisms of the derived functors.

6.1.9. *Semi-infinite standard resolutions.* Let $\mathfrak{a} = \bigoplus_{m\in\mathbf{Z}} \mathfrak{a}_m$ be a graded Lie algebra with a graded subalgebra $\mathfrak{b}$ such that $\mathfrak{b}_{>0}$ is of finite codimension in $\mathfrak{g}_{>0}$ and the 2-cocycle $\omega_0^{\mathfrak{b}} = 0$. Here we construct a semi-infinite analogue of the standard resolution 5.3.1. Recall the following statement crucial for Lie algebra semi-infinite cohomology.

6.1.10. **Proposition:** (see [V] 3.2.1, [Ar2] Corollary 4.4.2) Let $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathbf{C}K$ be the central extension of $\mathfrak{a}$ with the help of the cocycle $\omega_0^{\mathfrak{a}}$. Let $\widetilde{U(\mathfrak{a})} := U(\widetilde{\mathfrak{a}})/\{K-1\}$. Then there exists an inclusion of associative algebras $\widetilde{U(\mathfrak{a})} \hookrightarrow \mathrm{End}_{\mathfrak{a}}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}})$. □

We suppose below that the inclusion $\mathfrak{b} \subset \mathfrak{a}$ can be extended to $\mathfrak{b} \subset \widetilde{U(\mathfrak{a})}$.

For a graded vector space $V = \bigoplus_{m\in\mathbf{Z}} V_m$ consider the semi-infinite exterior powers

$$\Lambda^{\frac{\infty}{2}+\bullet}(V) := \Lambda(V^*_{>0}) \otimes \Lambda(V_{\leq 0}).$$

Lemma: (see e. g. [V] Proposition 2.3) Let V be a graded representation of a graded Lie algebra $\mathfrak{a}$. Then $\Lambda^{\frac{\infty}{2}+\bullet}(V)$ is a module over the central extension of $\mathfrak{a}$ with the help of ω_0^V. □

In particular the semi-infinite exterior powers of the $\mathfrak{b}$-module $\mathfrak{a}/\mathfrak{b}$ are $\mathfrak{b}$-modules as well.

6.1.11. Consider the Feigin standard complex $C^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}}) = S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+\bullet}(\mathfrak{a}) \otimes \underline{\mathbf{C}}$ for the computation of the semi-infinite Lie algebra cohomology of $\mathfrak{a}$ with coefficients in the semiregular $\mathfrak{a}$-module (using the second $\mathfrak{a}$-module structure on $S^{\mathfrak{a}}_{\mathfrak{a}_{>0}}$ from Proposition 6.1.10). Clearly we have

$$H^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}}) = \delta_{\bullet,0}\underline{\mathbf{C}}$$

and $C^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}}) \cong S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+\bullet}(\mathfrak{b}) \otimes \Lambda^{\frac{\infty}{2}+\bullet}(\mathfrak{a}/\mathfrak{b})$ as a vector space. We introduce a filtration of the complex as follows:

$$F^m(C^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}})) := \bigoplus_{\substack{p+q=n\\ q\geq m}} S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+p}(\mathfrak{b}) \otimes \Lambda^{\frac{\infty}{2}+q}(\mathfrak{a}/\mathfrak{b}).$$

Clearly the differential in the complex preserves the filtration. Consider the corresponding spectral sequence. We have

$$E_1^{p,q} = H^{\frac{\infty}{2}+p}(\mathfrak{b}, S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+q}(\mathfrak{a}/\mathfrak{b})) = \delta_{0,p} H^{\frac{\infty}{2}+0}(\mathfrak{b}, S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+q}(\mathfrak{a}/\mathfrak{b})).$$

Thus the spectral sequence converges in E_2. On the other hand $E_\infty^{p,q} = \delta_{p,0}\delta_{q,0}\underline{\mathbf{C}}$ and we obtain the following statement.

6.1.12. **Lemma:** There exists a complex of $\mathfrak{a}$-modules

$$\mathrm{St}_{\mathfrak{b}}^{\frac{\infty}{2}+\bullet} := H^{\frac{\infty}{2}+0}(\mathfrak{b}, S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+\bullet}(\mathfrak{a}/\mathfrak{b}))$$

such that $H^{\neq 0}(\mathrm{St}_{\mathfrak{b}}^{\frac{\infty}{2}+\bullet}) = 0,\ H^0(\mathrm{St}_{\mathfrak{b}}^{\frac{\infty}{2}+\bullet}) = \underline{\mathbf{C}}$. □

We call the obtained complex the *semi-infinite standard resolution* of the trivial $\mathfrak{a}$-module relative to the subalgebra $\mathfrak{b}$.

6.1.13. For a $\mathfrak{g}$-module M consider the complex of $\mathfrak{g}$-modules $\mathrm{St}_{\mathfrak{b}}^{\frac{\infty}{2}+\bullet} \otimes M$ (where the $\mathfrak{g}$-module structure uses the comultiplication on $U(\mathfrak{g})$). Evidently it is quasiisomorphic to M. We call the obtained complex the standard resolution of M and denote it by $\mathrm{St}_{\mathfrak{b}}^{\frac{\infty}{2}+\bullet}(M)$.

Let $\mathfrak{b}_1$, $\mathfrak{b}_2$ and $\mathfrak{b}$ be subalgebras in $\mathfrak{a}$ such that

$$\mathfrak{b} \subset \mathfrak{b}_1,\ \mathfrak{b} \subset \mathfrak{b}_2,\ \mathfrak{b}_{>0} = \mathfrak{b}_{1>0},\ \mathfrak{b}_{\leq 0} = \mathfrak{b}_{2\leq 0} \text{ and } \omega_0^{\mathfrak{b}_1} = \omega_0^{\mathfrak{b}_2} = \omega_0^{\mathfrak{b}} = 0.$$

6.1.14. **Lemma:** There exist natural quasiisomorphisms

$$\mathrm{St}_{\mathfrak{b}_2}^{\frac{\infty}{2}+\bullet}(M) \longrightarrow \mathrm{St}_{\mathfrak{b}}^{\frac{\infty}{2}+\bullet}(M) \longrightarrow \mathrm{St}_{\mathfrak{b}_1}^{\frac{\infty}{2}+\bullet}(M).$$

Proof. Recall that semi-infinite standard resolutions were constructed using levels E_1 of certain spectral sequences (see 6.1.11). Since $\mathfrak{b} \subset \mathfrak{b}_1$ and $\mathfrak{b}_{>0} = \mathfrak{b}_{1>0}$ we have

$$F^m_{\mathfrak{b}}(C^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}})) = \bigoplus_{\substack{p+q=n\\ q\geq m}} S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+p}(\mathfrak{b}) \otimes \Lambda^{\frac{\infty}{2}+q}(\mathfrak{a}/\mathfrak{b})$$

$$= \bigoplus_{\substack{p+q=n\\ q\geq m}} S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \bigoplus_{m+n=p} \Lambda^{-m}(\mathfrak{b}_{\leq 0}) \otimes \Lambda^n(\mathfrak{b}^*_{>0}) \otimes \bigoplus_{s+t=q} \Lambda^{-s}((\mathfrak{a}/\mathfrak{b})_{\leq 0}) \otimes \Lambda^t((\mathfrak{a}/\mathfrak{b})^*_{>0})$$

$$\subset \bigoplus_{\substack{p+q=n\\ q\geq m}} S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \bigoplus_{m+n=p} \Lambda^{-m}(\mathfrak{b}_{1\leq 0}) \otimes \Lambda^n(\mathfrak{b}^*_{1>0}) \otimes \bigoplus_{s+t=q} \Lambda^{-s}((\mathfrak{a}/\mathfrak{b}_1)_{\leq 0}) \otimes \Lambda^t((\mathfrak{a}/\mathfrak{b}_1)^*_{>0})$$

$$= F^m_{\mathfrak{b}_1}(C^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}})).$$

Since $\mathfrak{b} \subset \mathfrak{b}_2$ and $\mathfrak{b}_{\leq 0} = \mathfrak{b}_{2\leq 0}$ we have

$$F^m_{\mathfrak{b}_2}(C^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}})) = \bigoplus_{\substack{p+q=n\\ q\geq m}} S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \Lambda^{\frac{\infty}{2}+p}(\mathfrak{b}_2) \otimes \Lambda^{\frac{\infty}{2}+q}(\mathfrak{a}/\mathfrak{b}_2)$$

$$= \bigoplus_{\substack{p+q=n\\ q\geq m}} S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \bigoplus_{m+n=p} \Lambda^{-m}(\mathfrak{b}_{2\leq 0}) \otimes \Lambda^n(\mathfrak{b}^*_{2>0}) \otimes \bigoplus_{s+t=q} \Lambda^{-s}((\mathfrak{a}/\mathfrak{b}_2)_{\leq 0}) \otimes \Lambda^t((\mathfrak{a}/\mathfrak{b}_2)^*_{>0})$$

$$\subset \bigoplus_{\substack{p+q=n\\ q\geq m}} S^{\mathfrak{a}}_{\mathfrak{a}_{>0}} \otimes \bigoplus_{m+n=p} \Lambda^{-m}(\mathfrak{b}_{\leq 0}) \otimes \Lambda^n(\mathfrak{b}^*_{>0}) \otimes \bigoplus_{s+t=q} \Lambda^{-s}((\mathfrak{a}/\mathfrak{b})_{\leq 0}) \otimes \Lambda^t((\mathfrak{a}/\mathfrak{b})^*_{>0})$$

$$= F^m_{\mathfrak{b}}(C^{\frac{\infty}{2}+\bullet}(S^{\mathfrak{a}}_{\mathfrak{a}_{>0}})).$$

Thus we obtain the required morphisms of the spectral sequences that correspond to the morphisms of the filtered complexes. $\square$

6.2. **Twisted Verma modules and Wakimoto modules.** Consider the Lie algebra $\dot{w}(\mathfrak{b}^+)$ with the $\mathbf{Z}$-grading induced by the one on $\mathfrak{g}$ (see 5.1.4). Here we give a description of twisted Verma modules in terms of semi-infinite cohomology of $\dot{w}(\mathfrak{b}^+)$.

Consider the semiregular $U_k(\mathfrak{g})$-module with respect to $\mathfrak{n}^+$. The following statement is proved in [Ar2], 4.5.9. It is in fact a particular case of Proposition 6.1.10.

6.2.1. **Proposition:** $\mathrm{End}_{U_k(\mathfrak{g})}(S_{\mathfrak{n}^+}) \supset U_{-2h^\vee-k}(\mathfrak{g})$, where $h^\vee$ denotes the dual Coxeter number of $\overline{\mathfrak{g}}$. $\square$

Consider $S^{\mathfrak{g}}_{\mathfrak{n}^+}$ as a $\dot{w}(\mathfrak{b}^+)$-module using this second $\mathfrak{g}$-module structure.

6.2.2. **Lemma:**

$$H^{\frac{\infty}{2}+m}(\dot{w}(\mathfrak{b}^+), S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda)) = 0 \text{ when } m \neq 0,\ H^{\frac{\infty}{2}+0}(\dot{w}(\mathfrak{b}^+), S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda)) = S^{\dot{w}(\mathfrak{n}^-)}_{\dot{w}(\mathfrak{n}_w)} \otimes \mathbf{C}(\lambda)$$

as a $\mathfrak{h}^* \times \mathbf{Z}$-graded vector space. $\square$

Thus we have a $U_k(\mathfrak{g})$-module $\widetilde{M}_w(\lambda) := H^{\frac{\infty}{2}+0}(\dot{w}(\mathfrak{b}^+), S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda))$ that has the size of the twisted Verma module.

6.2.3. **Remark:** By definition of the $\mathfrak{g}$-module structure on $S^{\mathfrak{g}}_{\mathfrak{n}^+}$ the module $\widetilde{M}_w(\lambda)$ is cofree over $U(\dot{w}(\mathfrak{n}_w))$.

6.2.4. Consider also the Lie algebra $\mathfrak{b}^{\frac{\infty}{2}} := \overline{\mathfrak{h}} \otimes \mathbf{C}[t] \oplus \overline{\mathfrak{g}}^- \otimes \mathbf{C}[t, t^{-1}]$ and a $\mathfrak{h}^* \times \mathbf{Z}$-graded $U_k(\mathfrak{g})$-module $W(\lambda) := H^{\frac{\infty}{2}+0}(\mathfrak{b}^{\frac{\infty}{2}}, S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda))$.

Definition: The $\mathfrak{g}$-module $W(\lambda)$ is called the *Wakimoto module* at the level k of the weight λ.

We show that the modules $\widetilde{M}_w(\lambda)$ are indeed isomorphic to twisted Verma modules with the twist w and construct morphisms between them so that $W(\lambda)$ becomes their inductive limit as $\ell t(w)$ tends to infinity.

6.3. **Equivalence of definitions of twisted Verma modules.** Recall that a twisted Verma module with the highest weight λ is defined as $\Phi_w(M(w^{-1} \cdot \lambda))\langle \mathrm{ht}(w^{-1} \cdot \lambda - \lambda)\rangle$.

6.3.1. **Lemma:** $\widetilde{M}_w(\lambda) \cong M_w(\lambda)$.

Proof. Up to a $\mathbf{Z}$-grading shift we have

$$\Phi_w(M(w^{-1} \cdot \lambda)) = T_w \circ S_w(H^0(\mathfrak{b}^+, S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(w^{-1} \cdot \lambda)))$$
$$= H^0(\dot{w}(\mathfrak{b}^+), T_w((\mathrm{Coind}^{\mathfrak{g}}_{\dot{w}^{-1}(\mathfrak{n}^-)\cap\mathfrak{n}^-\oplus\mathfrak{h}} \mathrm{Ind}^{\dot{w}^{-1}(\mathfrak{n}^-)\cap\mathfrak{n}^-\oplus\mathfrak{h}}_{\mathbf{C}} \underline{\mathbf{C}}) \otimes \mathbf{C}(w^{-1} \cdot \lambda)))$$
$$= H^0(\dot{w}(\mathfrak{b}^+), S^{\mathfrak{g}}_{\dot{w}(\mathfrak{b}^+)\cap\mathfrak{n}^-} \otimes_{U_k(\mathfrak{g})} (S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda))) = H^0(\dot{w}(\mathfrak{b}^+), S^{\dot{w}(\mathfrak{b}^+)}_{\dot{w}(\mathfrak{b}^+)\cap\mathfrak{n}^-} \otimes_{U(\dot{w}(\mathfrak{b}^+))} (S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda))).$$

Now recall the abstract homological definition of Lie algebra semi-infinite cohomology from Remark 6.1.8. The $\dot{w}(\mathfrak{b}^+)$-module $S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda)$ is *semijective* (see [V] 3.1, [Ar1] Appendix B for the definition of semijectiveness). Thus we need no resolution to find the 2-sided derived functor:

$$H^0(\dot{w}(\mathfrak{b}^+), S^{\dot{w}(\mathfrak{b}^+)}_{\dot{w}(\mathfrak{b}^+)\cap\mathfrak{n}^-} \otimes_{U(\dot{w}(\mathfrak{b}^+))} (S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda))) = H^{\frac{\infty}{2}+0}(\dot{w}(b^+), S^{\mathfrak{g}}_{\mathfrak{n}^+} \otimes \mathbf{C}(\lambda)). \quad \square$$

6.3.2. **Proposition:**

$$\mathrm{St}^{\frac{\infty}{2}+\bullet}_{w(\mathfrak{b}^+)}(L(\lambda)) \cong \Phi_w(D^\bullet(L(\lambda))[-\ell t(w)]$$

where $[\cdot]$ denotes the shift of the homological grading.

Proof. The statement follows immediately from Lemmas 6.3.3 and 6.3.4. $\square$

6.3.3. **Lemma:** $\mathrm{St}^{\frac{\infty}{2}+\bullet}_{\mathfrak{b}^+} \cong D^\bullet$. $\square$

6.3.4. **Lemma:** $\Phi_w(\mathrm{St}_{\mathfrak{b}^+}^{\frac{\infty}{2}+\bullet}) \cong \mathrm{St}_{\dot{w}(\mathfrak{b}^+)}^{\frac{\infty}{2}+\bullet}[\ell t(w)]$.

Proof. The proof is parallel to the one of Lemma 6.3.1. □

The next statement shows that the twisted BGG resolutions are obtained from the semi-infinite standard resolutions relative to $\dot{w}(\mathfrak{b}^+)$ in the same way as the ordinary BGG resolution was obtained from the ordinary standard resolution relative to $\mathfrak{b}^+$ (see 5.3.1).

6.3.5. **Lemma:** For every $m \in \mathbf{Z}$ the $\mathfrak{g}$-module $\mathrm{St}_{\dot{w}(\mathfrak{b}^+)}^{\frac{\infty}{2}+\bullet}(L(\lambda))$ is finitely filtered by twisted Verma modules with the twist w. □

6.3.6. **Corollary:** $B_w^\bullet(\lambda) \cong \left(\mathrm{St}_{\dot{w}(\mathfrak{b}^+)}^{\frac{\infty}{2}+\bullet}(L(\lambda))\right)_{(\lambda+\rho,\lambda+\rho)}[\ell t(w)]$. □

6.4. **From twisted Verma modules to Wakimoto modules.** Fix $w_1, w_2 \in W$ such that $\ell t(w_2 w_1) = \ell t(w_1) + \ell t(w_2)$. Consider the following Lie subalgebras in $\mathfrak{g}$:

$$\mathfrak{b}_{w_2w_1,w_1} := \dot{w}_1(\mathfrak{b}^+) \cap \dot{w}_2\dot{w}_1(\mathfrak{b}^+),\ \mathfrak{b}_{w_2w_1,w_1}^{w_1} := \bigoplus_{\alpha \in R_{w_2w_1,w_1}^{w_1}} \mathfrak{g}_\alpha,\ \mathfrak{b}_{w_2w_1,w_1}^{w_2w_1} := \bigoplus_{\alpha \in R_{w_2w_1,w_1}^{w_2w_1}} \mathfrak{g}_\alpha,$$

where $R_{w_2w_1,w_1}^{w_1} := \{\alpha \in w_1(R^+) | w_2(\alpha) \notin w_1(R^+)\}$, $R_{w_2w_1,w_1}^{w_2w_1} := \{\alpha \in w_2w_1(R^+) | w_1^{-1}(\alpha) \notin w_1(R^+)\}$.

Then $\mathfrak{b}_{w_2w_1,w_1} \oplus \mathfrak{b}_{w_2w_1,w_1}^{w_1} = \dot{w}_1(\mathfrak{b}), \mathfrak{b}_{w_2w_1,w_1} \oplus \mathfrak{b}_{w_2w_1,w_1}^{w_2w_1} = \dot{w}_2\dot{w}_1(\mathfrak{b})$. By the choice of w_1 and w_2 we have $\mathfrak{b}_{w_2w_1,w_1}^{w_1} \subset \mathfrak{n}^+$ and $\mathfrak{b}_{w_2w_1,w_1}^{w_2w_1} \subset \mathfrak{n}^-$. In particular using 6.1.7 one can define morphisms of $\mathfrak{g}$-modules

$$H^{\frac{\infty}{2}+\bullet}(\dot{w}_1(\mathfrak{b}^+), S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)) \longrightarrow H^{\frac{\infty}{2}+\bullet}(\mathfrak{b}_{w_2w_1,w_1}, S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)) \longrightarrow H^{\frac{\infty}{2}+\bullet}(\dot{w}_2\dot{w}_1(\mathfrak{b}^+), S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)),$$

i. e. $M_{w_1}(\lambda) \longrightarrow M_{w_2w_1}(\lambda)$. Denote these morphisms by $\varphi_{w_2w_1,w_1}^\lambda$.

For $w \in W$ such that $\dot{w}(\mathfrak{b}^+)_{\leq 0} \cap \mathfrak{b}^{\frac{\infty}{2}} = \dot{w}(\mathfrak{b}^+)_{\leq 0}$ and $\dot{w}(\mathfrak{b}^+) \cap \mathfrak{b}_{>0}^{\frac{\infty}{2}} = \mathfrak{b}^{\frac{\infty}{2}}$ consider maps

$$H^{\frac{\infty}{2}+\bullet}(\dot{w}(\mathfrak{b}^+), S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)) \longrightarrow H^{\frac{\infty}{2}+\bullet}(\dot{w}(\mathfrak{b}^+) \cap \mathfrak{b}^{\frac{\infty}{2}}, S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)) \longrightarrow H^{\frac{\infty}{2}+\bullet}(\mathfrak{b}^{\frac{\infty}{2}}, S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)),$$

i. e. $M_w(\lambda) \longrightarrow W(\lambda)$. Denote these morphisms by $\varphi_{\frac{\infty}{2},w}^\lambda$.

6.4.1. **Lemma:**

(i) For $w_1, w_2, w_3 \in W$ such that $\ell t(w_3w_2w_1) = \ell t(w_1) + \ell t(w_2) + \ell t(w_3)$ we have

$$\varphi_{w_3w_2w_1,w_2w_1}^\lambda \circ \varphi_{w_2w_1,w_1}^\lambda = \varphi_{w_3w_2w_1,w_1}^\lambda.$$

(ii) For $w_1, w_2 \in W$ such that $\ell t(w_2w_1) = \ell t(w_1) + \ell t(w_2)$ we have $\varphi_{\frac{\infty}{2},w_2w_1}^\lambda \circ \varphi_{w_2w_1,w_1}^\lambda = \varphi_{\frac{\infty}{2},w_1}^\lambda$. □

Fix $w_1, w_2 \ldots \in W$ such that for each m we have $\ell t(w_m \ldots w_1) = \ell t(w_1) + \ldots + \ell t(w_m)$. Thus for each λ we obtain an inductive system consisting of twisted Verma modules with twists $w_1, w_2w_1, \dots \in W$ and the lengths of $w_m \ldots w_1$ tending to infinity. Suppose the chosen sequence of elements of the Weyl group belongs to the image of $-Q''^+$ in T. Then we obtain the following statement.

6.4.2. **Lemma:** $W(\lambda) = \varinjlim M_{w_m \ldots w_1}(\lambda)$.

Proof. The maps $\varphi_{w_m \ldots w_1, w_{m-k} \ldots w_1}^\lambda$ are well defined as maps

$$C^{\frac{\infty}{2}+\bullet}(\dot{w}_{m-k} \ldots \dot{w}_1(\mathfrak{b}^+), S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)) \longrightarrow C^{\frac{\infty}{2}+\bullet}(\dot{w}_m \ldots \dot{w}_1(\mathfrak{b}^+), S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)).$$

On the other hand we have

$$C^{\frac{\infty}{2}+\bullet}(\mathfrak{b}^{\frac{\infty}{2}}, S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)) = \varinjlim C^{\frac{\infty}{2}+\bullet}(\dot{w}_m \ldots \dot{w}_1(\mathfrak{b}^+), S_{\mathfrak{n}^+}^{\mathfrak{g}} \otimes \mathbf{C}(\lambda)).$$

But the direct image functor is exact. □

We are almost done. It remains to prove that roughly speaking the marphisms $\varphi_{w_2w_1,w_1}^\lambda$ indeed form an inductive system of complexes (twisted BGG resolutions). The combinalorial proof of this statement requires the following fact:

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M_w(\mu)) \leq 1.$$

Unfortunately I do not know any proof of the fact, and our considerations become more cumbersome.

6.5. **The limit procedure.** Fix the sequence $w_1, \dots, w_m, \dots \in W$ satisfying the conditions from the previous paragraph. Consider the standard resolutions $\mathrm{St}^{\frac{\infty}{2}+\bullet}_{\dot{w}_m \dots \dot{w}_1(\mathfrak{b}^+)}(L(\lambda))$ (see 6.1.13). Then for every m and k the algebras $\dot{w}_{m+k} \dots \dot{w}_1(\mathfrak{b}^+)$, $\dot{w}_m \dots \dot{w}_1(\mathfrak{b}^+)$ and $\dot{w}_{m+k} \dots \dot{w}_1(\mathfrak{b}^+) \cap \dot{w}_m \dots \dot{w}_1(\mathfrak{b}^+)$ satisfy the conditions of Lemma 6.1.14, and we obtain canonical morphisms of complexes

$$\psi^\lambda_{w_{m+k}\dots w_1, w_m \dots w_1} : \mathrm{St}^{\frac{\infty}{2}+\bullet}_{\dot{w}_m \dots \dot{w}_1(\mathfrak{b}^+)}(M) \longrightarrow \mathrm{St}^{\frac{\infty}{2}+\bullet}_{\dot{w}_{m+k} \dots \dot{w}_1(\mathfrak{b}^+) \cap \dot{w}_m \dots \dot{w}_1(\mathfrak{b}^+)}(M) \longrightarrow \mathrm{St}^{\frac{\infty}{2}+\bullet}_{\dot{w}_{m+k} \dots \dot{w}_1(\mathfrak{b}^+)}(M).$$

Thus we obtain an inductive system complexes, and all morphisms of complexes in it are quasiisomorphisms by 6.1.14.

6.5.1. **Theorem:** There exists a complex in $\mathcal{M}_k$ quasiisomorphic to $L(\lambda)$ of the form

$$\dots \longrightarrow \bigoplus_{\substack{w \in W, \\ \ell t^{\frac{\infty}{2}}(w) = m}} W(w \cdot \lambda)\langle -\operatorname{ht}(\lambda - w \cdot \lambda)\rangle \longrightarrow \dots \longrightarrow \bigoplus_{\substack{w \in W, \\ \ell t^{\frac{\infty}{2}}(w) = 0}} W(w \cdot \lambda)\langle -\operatorname{ht}(\lambda - w \cdot \lambda)\rangle \longrightarrow$$
$$\bigoplus_{\substack{w \in W, \\ \ell t^{\frac{\infty}{2}}(w) = -1}} W(w \cdot \lambda)\langle -\operatorname{ht}(\lambda - w \cdot \lambda)\rangle \longrightarrow \dots \bigoplus_{\substack{w \in W, \\ \ell t^{\frac{\infty}{2}}(w) = -m}} W(w \cdot \lambda)\langle -\operatorname{ht}(\lambda - w \cdot \lambda)\rangle \longrightarrow \dots$$

Proof. We know already from Corollary 6.3.6 that when restricted to the Γ-homogenious direct summands with eigenvalue of Γ equal to $(\lambda+\rho, \lambda+\rho)$ the morphisms

$$\psi^\lambda_{w_{m+k}\dots w_1, w_m \dots w_1} : \mathrm{St}^{\frac{\infty}{2}+\bullet}_{\dot{w}_m \dots \dot{w}_1(\mathfrak{b}^+)}(L(\lambda)) \longrightarrow \mathrm{St}^{\frac{\infty}{2}+\bullet}_{\dot{w}_{m+k} \dots \dot{w}_1(\mathfrak{b}^+)}(L(\lambda))$$

form an inductive system of morphisms between twisted BGG resolutions

$$\psi^\lambda_{w_{m+k}\dots w_1, w_m \dots w_1} : \mathcal{B}^\bullet_{w_m \dots w_1}(\lambda) \longrightarrow \mathcal{B}^\bullet_{w_{m+k} \dots w_1}(\lambda)[-\ell t(w_{m+1}) - \dots - \ell t(w_{m+k})].$$

Now we choose a sequence of elements of the affine Weyl group satisfying the conditions from 6.4.1 and belonging to the image of $-Q'^{+}$ in $T \subset W$. Thus by Lemma 4.2.7 for any $w \in W$ there exists $m_0 \in \mathbf{N}$ such that for any $m \geq m_0$

$$M_{w_m \dots w_1}(w \cdot \lambda)\langle -\operatorname{ht}(\lambda - w \cdot \lambda)\rangle \hookrightarrow B^{\ell t^{\frac{\infty}{2}}(w)}_{w_m \dots w_1}(\lambda)[-\ell t(w_m) - \dots - \ell t(w_1)].$$

It remains to check that for every $w \in W$ and m sufficiently large we have

$$\psi^\lambda_{w_{m+k}\dots w_1, w_m \dots w_1}|_{M_{w_m \dots w_1}(w \cdot \lambda)} = \varphi^{w \cdot \lambda}_{w_{m+k}\dots w_1, w_m \dots w_1}.$$

□

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